Wave-driven vortex dynamics in the surf zone

Oliver Bühler
Tivon Jacobson, Andrea Barreiro

Center for Atmosphere Ocean Science
Interdisciplinary PhD program in Atmosphere Ocean Science and Mathematics

Courant Institute of Mathematical Sciences
New York University
Wave-driven currents and vortex dynamics on barred beaches

By OLIVER BÜHLER\textsuperscript{1} and TIVON E. JACOBSON\textsuperscript{2}

\textsuperscript{1}School of Mathematics and Statistics, University of St Andrews, St Andrews KY16 9SS, UK
\textsuperscript{2}Courant Institute of Mathematical Sciences, New York, NY University, New York, NY 10012, USA

(Received 26 October 2000 and in revised form 21 May 2001)
Emission of deep-water gravity waves

Dispersion: \[ \omega = \sqrt{gk} \]

approx. 15km/h for 50m wavelength

Dispersion generates slowly varying wavetrain
Longshore currents

Incoming waves

Surf zone

Beach

Wave-induced momentum flux into surf zone

Absorption due to wave breaking

Momentum flux convergence drives longshore current

Topography

\[ h - h_S = \text{surface elevation} \]
Shallow water model

\[ \frac{\partial h}{\partial t} + \nabla \cdot (hu) = 0 \]

Two-dimensional domain \( x = (x, y), \quad u = (u, v) \)

Mass
\[ \frac{\partial h}{\partial t} + \nabla \cdot (hu) = 0 \]

Momentum
\[ \frac{Du}{Dt} + g \nabla (h - h_S) = -c_f \frac{|u|}{h} + F \]
Assumed alongshore uniform wavetrain and topography, i.e. \( h_S(x) \) only

Small angle of incidence: wavenumber vector \( k = (k, l) \) with \( k^2 \gg l^2 \)

Ray tracing theory in cross-shore direction \( x \)

\[
l = l_0, \quad \omega^2 = \omega_0^2 \implies k^2 h_S(x) = k_0^2 h_S(x_0)
\]

(in simplest, shallow water regime)

\[
E = \frac{1}{2} \left( \overline{u'^2} + \overline{v'^2} + PE \right)
\]

Wave action conservation \( \frac{\partial}{\partial x} (h_S E u_g) = 0 \)

Small amplitude waves such that \( h'/h_S = O(a), \quad a \ll 1 \)

Unbounded inviscid growth: waves must break

\[
E(x) \propto h_S^{-3/2}
\]
**LH 70: mean momentum budget**

\[
\langle \ldots \rangle = \text{average over one wave period}
\]

\[
u = \bar{u} + u' \quad \text{and} \quad \bar{u}' = 0 \quad \text{etc.}
\]

Steady, alongshore homogeneous: \( \bar{u} = 0, \bar{v}(x) \)

**Alongshore momentum budget:**

\[
\frac{\partial \bar{v}}{\partial t} + \frac{1}{h_S} \frac{\partial}{\partial x} (h_S \bar{u}'\bar{v}') = 0
\]

\( \bar{v} \) = 0  

`radiation stress convergence`

\[
\nabla \cdot \left( \bar{u}' \bar{f}' \right) = \text{quadratic bottom friction} \propto -\bar{v} \sqrt{E} + \text{(horizontal “eddy” diffusion)}
\]

Linear waves with amplitude
imply radiation stress
Bottom friction
and hence balance requires

\[
a \ll 1
\]

\[
h_S \bar{u}'\bar{v}' = O(a^2)
\]

\[
\propto \bar{v} \sqrt{E} \propto \bar{v}O(a)
\]

Saturated mean flow same order as waves; saturation time:

\[
T = O(a^{-1})
\]
LH 70: momentum flux outside surf zone

\[ h_S \overline{u'v'} = h_S \frac{kl}{k^2 + l^2} E \propto h_S^{3/2} E = \]

constant if \[ E(x) \propto h_S^{-3/2} \]

Constant flux without wave breaking:

- typical inviscid non-acceleration result
LH 70: surf zone model

In surf zone wave breaks and wave amplitude saturates

Heuristic saturation model: \( h' \propto h_S \Rightarrow E \propto h_S \)

Radiation stress: \( h_S u'v' \propto h_S^{3/2} E \propto h_S^{5/2} \)

hence momentum flux decreases towards shoreline

Net alongshore driving force balanced by bottom friction

Longshore current is strongest where wave breaking strongest
LH theory performance and beach shape

- good on **planar** beaches
- not so good on **barred** beaches

Dislocation effect tends to be stronger for stronger waves and high tides
### Additions to LH70 1d theory

<table>
<thead>
<tr>
<th>Description</th>
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<tbody>
<tr>
<td>Stochastic waves, Boussinesq wave transformation models more realistic waves and fluctuations, but no current displacement</td>
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<tr>
<td>Different kinds of bottom friction, or of horizontal eddy diffusion, small effects</td>
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<tr>
<td>Ad hoc delay of momentum flux convergence: “wave rollers” best practical success to date, produces current displacement, but ad hoc closure. Tuning problem (e.g. Goda 2006).</td>
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Fundamental scientific question:

Why does LH70 work for planar beaches but not for barred beaches?

A good theory should answer this question!

Ad hoc fixes for barred beaches offer no good answers to this question.
Outside LH 70: vortices in the surf zone

Vortex generators

Mean flow instabilities

Alongshore inhomogeneities in topography and/or wave fields

Vortex effects

Lateral mixing, rip currents

Displacement of mean current location
Local wave breaking creates vortex dipoles e.g. non-uniform shocks in shallow water Bühler 2000

Wavepacket

Vortex dipole

dissipation

Important for wave-mean interactions e.g. Bühler 2000, Bühler & McIntyre, J.Fluid Mech., 2003, 2005

Role of wave-driven vortices in the surf zone?
Case for essential 2d modelling

(Bühler & Jacobson, JFM 2001)

Momentum transport by 2d vortex motion is not diffusive

Can obtain current displacement without rollers
2d vorticity view

\[ \frac{\partial}{\partial t} \Delta h + \nabla \cdot (h_S \bar{u}^L) = 0 \]

Lagrangian-mean velocity includes Stokes drift

\[ \frac{\partial}{\partial t} \bar{u}^L + g \nabla \Delta h = -\frac{1}{h_S} \nabla \cdot S \]

Pseudomomentum (Stokes drift)

\[ p = \frac{1}{h_S} h' u' \]

Dissipative force related to energy dissipation \( D \)

\[ \mathcal{F} = \frac{k}{\hat{\omega}} D \]

Example: two-dimensional wave `set-up' without dissipation \( \Delta h \propto -|u'|^2 \)

Example: vorticity generation

\[ \frac{\partial}{\partial t} (\nabla \times [\bar{u}^L - p]) = \nabla \times \mathcal{F} \]

Contains LH70 as special case: steady, y-independent waves (Disagrees with Putrevu, Oltman-Shay, Svendsen 1995)
Vortex dynamics on a sloping beach

Shoreline approach and separation

\[ \Gamma > 0 \]
\[ \Gamma < 0 \]
\[ 2b \]
\[ d \]
\[ \Gamma < 0 \]
\[ \Gamma > 0 \]

Lateral motion due to vortex curvature

\[ U \propto \frac{\Gamma}{h_S} (\nabla h_S \times \hat{z}) \]
Vortex pair climbing a planar beach

\[ \mathbf{U} \propto \frac{1}{h_s} (\nabla h_s \times \mathbf{\hat{z}}) \rightarrow \text{Vortex couples move into opposite (alongshore) direction.} \]

\[ \nabla h_s = \leftarrow \]

\[ h_s = \text{cont.} \]

\[ \text{Vortices separate when moving into shallower water. To slow-down.} \]
Vortex pair leaving a planar beach

Vortices get closer when moving into deeper water. [rip current]

"Vortex couples alike deep water"... a deep truth...
Planar & barred beaches

(i) planar beach: $h_{s1x} < 0$ everywhere.

- vortices quickly separate along the shoreline due to both self-advection & image vortices.

- expect very little current dislocation

(ii) barred beach: $h_{s1x} > 0$ in places.

- vortices move rapidly into the deep troughs, then separate on the shoreward climb.

- expect current dislocation crest-to-trough

Offers an answer to the fundamental scientific question!
Numerical set up of shallow water model

Wave-resolving shock-capturing two-dimensional finite-volume shallow-water model

Penalty: need small time step & low wave amplitude to prevent premature breaking

$a \approx 0.015$
Model topography for barred beach

Crest 100m

Trough 150m

$h_B$

$x$
Snapshots of surface undulation

Figure 1:

homogeneous

inhomogeneous

Video now!
Surface undulation
Potential vorticity - no bottom friction

\[ pv = \nabla \times \frac{u}{h} \]
Potential vorticity - with bottom friction
Longshore current

\[ \bar{v}(x, t) = \frac{1}{L} \int_0^L v \, dy \]
Longshore currents comparison - with bottom friction
**Take home**

<table>
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<th>Proposed vortex dynamics gives robust mechanism for current dislocation</th>
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<td>Same mechanism offers explanation why this happens strongly on barred beaches and weakly on planar beaches</td>
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Shallow water model drawbacks

Shallow-water model very diffusive (truncation error) especially in vortex dynamics

Wave amplitude very restricted

Long integrations costly

Build new model with very simple wave parametrization and focus on vortex dynamics with very low diffusion
New numerical model: PV-Beach
Andrea Barreiro

Vorticity-based model for rigid-lid flow with parametrized wave forcing

2d ray tracing & saturation to determine $\nabla \times \mathcal{F}$

Fully resolved 2d dynamics, no eddy diffusivity

$$\nabla \cdot (hSu) = 0$$

$$q = \frac{\nabla \times u}{h_S}$$

$$\frac{Dq}{Dt} = \frac{\nabla \times (F)}{h_S}$$

Mass stream function

$$\psi_x = +h_S v$$

$$\psi_y = -h_S u$$

$$\nabla \cdot \left( \frac{\nabla \psi}{h_S} \right) = h_S q$$

Time-stepping: invert PV to find stream function, evolve PV, repeat. Advection discretized with Arakawa Jacobian such that truncation error is purely dispersive: energy & enstrophy conserved
**PV-Beach simulation**

Barred beach with crest at 100m, trough at 50m

Domain 512x512m, periodic in alongshore direction

Incoming waves 15 degrees, amplitude 0.2

Gaussian wavetrain envelope

Quadratic bottom friction with coefficient 0.01

Shore on the left!
Curl of wave forcing

Ray tracing and saturation gives $\nabla \times \mathcal{F}$

Barred beach with crest at 100m, trough at 50m

Shore on left!
Vorticity snapshot

Vorticity field at $t[s] = 20000$
Varying wavetrain width

With constant net momentum flux the longshore current is only weakly dependent on wavetrain width

Narrower 50%  Wider 50%

Typically find laminar, nearly steady vorticity structures at long time. Is two-dimensional turbulence not relevant?

Enhanced nearest neighbour interaction
Stronger rip current
"Anti-turbulence" result for two-dimensional turbulence on a beach

Turbulence studies with quadratic bottom friction show that energetic turbulence is confined to horizontal wavenumbers above frictional arrest wavenumber

\[ k_a \approx 40 \frac{c_f}{h} \]

Typical energy cascade in two-dimensional turbulence (Grianik et al. 2003)

Studies are done with constant water depth, but can be applied to the beach

Is vigorous turbulence possible on a beach?
No vigorous turbulence on shallow water beach..

\[ k_a \approx 40 \frac{c_f}{h} \approx 0.4 \frac{1}{h} \]

Using typical order of magnitude for friction coefficient on beaches

Most dynamics in shallow water is at or below arrest wavenumber: laminar, quasi-periodic.
Consistent with PV-Beach simulations.
Conclusions

Wave-driven 2d vortex dynamics is non-diffusive and should be resolved

Vortices like deep water and can displace currents into the trough by rip current mechanism

Vigorous 2d turbulence is unlikely on beach due to strong bottom friction

PV-Beach model for 2d topography
Multigrid solver
Open ocean boundary condition instead of solid wall

Comparison with (1d) lab data
Hamilton & Ebersole (2001) and with (2d) DELILAH 1990 field experiment data

“adapted” from http://www.onr.navy.mil/focus/ocean/motion/currents2.htm