Buoyancy Driven Flows

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Gravity flow on steep slope

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1.1 Introduction

Particle-laden gravity-driven flows occur in a large variety of natural and industrial situations. Typical examples include turbidity currents, volcanic eruptions, and sand storms (see Simpson, 1997, for a review). On mountain slopes, debris flows and snow avalanches provide particular instances of vigorous dense flows, which have special features that make them different from usual gravity currents:

- they belong to the class of non-Boussinesq flows since the density difference between the ambient fluid and the flow is usually very large while most gravity currents are generated by a density difference of a few percent;
- while many gravity currents are driven by pressure gradient and buoyancy forces, the dynamics of flows on slope are controlled by the balance between the gravitational acceleration and dissipation forces. Understanding the rheological behavior of particle suspensions is often of paramount importance when studying gravity flows on steep slope.

This chapter reviews some of the essential features of snow avalanches and debris flows. Since these flows are a major threat to human activities in mountain areas, they have been studied since the late 19th century. In spite of the huge amount of work done in collecting field data and developing flow-dynamics models, there remain great challenges in efforts to understand the dynamics of flows on steep slope and, ultimately, predict their occurrence and behavior. Indeed, these flows involve a number of complication such
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as abrupt surge fronts, varying free and basal surfaces, flow structure that changes with position and time.

Subaqueous landslides and debris avalanches present many similarities with subaerial debris flows and avalanches, and thus can benefit from their study (Hampton et al., 1996). The correspondence is, however, not complete since subaqueous debris flows are prone to hydroplane and transform into density currents as a result of water entrainment (Elverhøi et al., 2005); the slope range over which they occur is also much wider than for subaerial flows. Powder snow avalanches are related to turbidity currents on the ocean floor (Parker et al., 1986) and pyroclastic flows from volcanoes (Huppert & Dade, 1998). Powder snow avalanches sometimes experience a rapid deceleration of their dense cores, which eventually separate from their dilute clouds and form stepped thickness patterns in their deposits. This behavior is also seen with submarine flows and pyroclastic flows. In addition to being non-Boussinesq flows, powder-snow avalanches differ from submarine avalanches in that they are closer to fixed-volume, unsteady currents rather than the steady density currents with constant supply.

1.2 A physical picture of gravity flows

1.2.1 Debris flows

Debris flows are mass movements of concentrated slurries of water, fine solids, rocks and boulders (Iverson, 1997, 2005). They are highly concentrated mixtures of sediments and water, flowing as a single-phase system on the bulk scale. Debris flows look like mudslides and landslides except that their velocity and the distances they travel are much larger. They differ from floods with sediment transport in that they are characterized by a very high solids fraction (mostly exceeding 80%).

There are many classifications of debris flows and related phenomena, which are based on the compositions, origins, and appearances. Many events categorized as mudflows, debris slides, lahars, hyperconcentrated flows can be considered as particular forms of debris flows (Fannin & Rollerson, 1993; Iverson, 1997). Debris flows may result from:

- mobilization from a landsliding mass of saturated unsorted materials, often after heavy and/or sustained rainfalls (Iverson et al., 1997);
- transformation from a sediment-laden water flood into a hyperconcentrated flow, probably as a result of channel-bed failure (Tognacca, 1997);
- melting of ice and snow induced by pyroclastic or lava flows and accompanied by entrainment of large ash volumes (Voight, 1990);
1.2 A physical picture of gravity flows

Figure 1.1 (a) Debris-flow deposit in the Ravin-des-Sables watershed (France); the bucket gives a scale of the deposit thickness. (b) Debris flow on the road to la-Chapelle-en-Valgaudemar (France).

- collapse of a moraine-dammed lake generating an outburst flood (Clague & Evans, 2000).

The material volume mobilized by debris flows ranges from a few thousands cubic meters to a few millions, exceptionally a few billions. The velocity is typically of a few meters per second, with peak velocities as high as 10 m/s (VanDine, 1985; Major & Pierson, 1992; Hürlimann et al., 2003). Debris flows usually need steep slope (i.e., in excess of 20%) to be initiated and flow, but occasionally, they have been reported to travel long distances over shallow slopes (less than 10%).

Figure 1.1 shows two deposits of debris flows. In Fig. 1.1(a), a debris flow involving well-sorted materials embedded in a clayey matrix came to a halt on an alluvial fan; note that there was no water seepage, which implies that the material was still water-saturated a few hours after stoppage. Figure 1.1(b) shows a car hit by a debris flow made up of coarse material; the conspicuous streaks of muddy water indicate that water and the finest grain fraction separated from the coarsest grain fraction as soon as the flow approached the arrested state.

1.2.2 Snow avalanches

Avalanches are rapid gravity-driven masses of snow moving down mountain slopes. Many, if not most, catastrophic avalanches follow the same basic principle: fresh snow accumulates on the slope of a mountain until the gravitational force at the top of the slope exceeds the binding force holding the snow together. A solid slab of the surface layer of snow can then push its way across the underlying layer, resulting in an avalanche. The failure may also
arise from a temperature increase, which reduces snow cohesion. Typically, most avalanches travel for a few hundred meters at a rather low velocity (a few meters per second), but some can move up to 15 km and achieve velocities as high as 100 m/s. They can also pack an incredible punch, up to several atmospheres of pressure. It is helpful to consider two limiting cases of avalanches depending on the flow features (de Quervain, 1981):

- The flowing avalanche: a flowing avalanche is an avalanche with a high-density core at the bottom. Trajectory is dictated by the relief. The flow depth does not generally exceed a few meters [see Fig. 1.2(a)]. The typical mean velocity ranges from 5 m/s to 25 m/s. On average, the density is fairly high, generally ranging from 150 kg/m$^3$ to 500 kg/m$^3$.

- The powder snow avalanche: it is a very rapid flow of a snow cloud, in which most of the snow particles are suspended in the ambient air by turbulence [see Fig. 1.2(b)]. Relief has usually weak influence on this aerial flow. Typically, for the flow depth, mean velocity, and mean density, the order of magnitude is 10–100 m, 50–100 m/s, 5–50 kg/m$^3$ respectively.

Figure 1.2 (a) Wet-snow-avalanche deposit in the southern face of Grammont (Switzerland); the snowballs are approx. 10 cm in diameter. (b) Powder-snow avalanche in the northern face of Dolent (Switzerland); the typical flow depth is 20 m.

1.3 Anatomy of gravity currents on slope

How a gravity-driven flow is organized is of paramount importance to understanding its properties. Contrary to most fluid-mechanics problems—in which the fluid volume is bounded or infinite—, a gravity current is characterized by moving boundaries:

- the free surface at the interface with the ambient air;
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• the surface of contact with the ground (or snow cover), where much of the energy dissipation occurs.

These boundaries can be passive, i.e. they mark the boundaries of the volume occupied by the flowing material. On some occasions, they may be active, e.g., by promoting mass and momentum exchanges with the ambient fluid and/or the bed. A gravity-driven flow is also often split into three parts: the head at the leading edge, the body, and the tail. The structure of these regions depend on the material and flow properties. It is quite convenient to consider two end members of gravity flows to better understand their anatomy: debris flows are typical of dense granular flows, for which the ambient fluid has no significant dynamic role while powder-snow avalanches are typical of flows whose dynamics are controlled to a large extent by the mass and momentum exchanges at the interfaces.

1.3.1 Anatomy of debris flows

On the whole, debris flows are typically characterized by three regions, which can change with time (see Fig. 1.3):

• At the leading edge, a granular front or snout contains the largest concentration of big rocks. Boulders seem to be pushed and rolled by the body of the debris flow. The front is usually higher than the rest of the flow. In some cases no front is observed because either it has been overtaken by the body (this is very frequent when the debris flow spreads onto the alluvial fan), or the materials are well sorted and no significant variation in the bulk composition can be detected.

• Behind the front, the body has the appearance of a more fluid flow of a rock and mud mixture. Usually, the debris flow body is not in a steady state but presents unsteady surges (Zamuttigh & Lamberti, 2007). It can transport blocks of any size. Many authors have reported that boulders of relatively small size seem to float at the free surface while blocks of a few meters in size move merely by being overturned by the debris flow. The morphological characteristics of the debris flow are diverse depending on debris characteristics (size distribution, concentration, mineralogy) and channel geometry (slope, shape, sinuosity, width). Flowing debris can resemble wet concrete, dirty water, or granular material but irrespective of the debris characteristics and appearance, viscosity is much higher than for water. Most of the time, debris flows move in a completely laminar fashion, but they can also display minor turbulence; on some occasions, part of the debris flow may be highly turbulent.
In the tail, the solid concentration decreases significantly and the flow looks like a turbulent muddy water flow.

Figure 1.3 Idealized representations of a debris flow (longitudinal profile and cross section). The different sections correspond to the dashed lines of the upper panel. Adapted from (Johnson & Rodine, 1984).

In recent years, many outdoor and laboratory experiments have shed light on the connections existing between particle-size distribution, water content, and flow features for fixed volumes of bulk material (Davies, 1986; Iverson, 1997; Parsons et al., 2001; Chambon et al., 2009). In particular, experiments performed by Parsons et al. (2001) and Iverson (1997) have shown that the flow of poorly sorted materials was characterized by the coexistence of two zones, each one with a distinctive rheological behavior: the flow border was rich in coarse-grained materials, while the core was fine-grained. This self-organization has a great influence on the flow behavior; notably, the flow core behaves more like a viscoplastic material, while the flow region close to the levees are in a Coulomb frictional regime (sustained solid frictional contacts between grains). Moreover, the run-out distance can be significantly enhanced as a result of levee formation limiting lateral spreading and energy dissipation.

Parsons et al. (2001) ran a series of experiments to investigate the effect of the composition (i.e. the importance of the finest- and coarsest-grain fractions). They used a semi-circular inclined flume and measured the velocity profile at the free surface. Different slurries were prepared by altering the sand, clay, and silt fractions. They obtained muddy slurries, when the matrix
was rich in silt and clay, and poorly sorted mixtures, when the silt and clay contents were reduced. Surprisingly enough, the change in the fine-particle content did not significantly modify the appearance of the body, whereas it markedly altered the composition of the front and its behavior. Reducing the fine fraction in the slurries induced a radical change of behavior for the front (see Fig. 1.4):

- For muddy slurries, the front took the form of a blunt nose. Lack of slip along the flume bottom caused a conveyer-belt-like flow at the front.
- For coarse-grained slurries, the front took the form of a dry granular locked nose slipping along the bed as a result of the driving force exerted by the fluid accumulating behind the snout. Additional material was gradually incorporated into the snout, which grew in size until it was able to slow down the body.

Interestingly enough, the changes in the rheological properties mainly affected the structure of the flow, especially within the tip region.

Iverson and his colleagues investigated slurries predominantly made up of a water-saturated mixture of sand and gravel, with a fine fraction of only a few percent (Iverson, 1997, 2003a, 2005; Iverson et al., 2010). Experiments were run by releasing a volume of slurry (approximately 10 m$^3$) down a 31-degree, 95-m-long flume. At the base of the flume, the material spread out on a planar, nearly horizontal, unconfined runout zone. Flow-depth, basal normal stress, and basal interstitial-flow pressure were measured at different places along the flume. Iverson and his co-workers observed that at early times, an abrupt front formed at the head of the flow, followed by a
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gradually tapering body, then a thin, more watery tail. The front remained relatively dry (with pore pressure dropping to zero) and of constant thickness, while the body elongated gradually in the course of the flow. Over the longest part of the flume, the basal pore pressure nearly matched the total normal stress, which means that shear strength was close to zero and the material was liquefied within the body (Iverson, 1997). In their recent data compilation, Iverson et al. (2010) confirmed the earlier observations made by Parsons et al. (2001): mud enhanced flow mobility by maintaining high pore pressures in flow bodies. They also observed that roughness reduced flow speeds, but not runout distances. The explanation for this apparently strange behavior lies in the particular role played by debris agitation and grain-size segregation. Indeed, if the bed is flat, particles slip along the bottom and shear is localized within a thin layer close to the bed, with almost no deformation through the flow depth (i.e. uniform velocity profile). In contrast, if the bed is corrugated, particles undergo collisions and are more agitated, which promotes the development of a nonuniform velocity profile through the depth and causes the flow to slightly dilate. Velocity shear and dilatancy act together as a random fluctuating sieve that allows the finer particles to percolate to the bottom under the action of gravity, while squeezing larger particles upwards. This segregation process may have a feedback on the bulk flow, e.g., by reducing bottom friction during the course of motion (Linares-Guerrero et al., 2007; Gray, 2010).

Figure 1.5 shows a sequence of aerial photographs taken when the material spread out on the runout surface. Self-organization of the slurry flow into a coarse-grained boundary and a muddy core became conspicuous as the flow travelled the runout surface. Lateral levees were formed by the coarsest grains that reach the front, being continuously shouldered aside by the muddy core. These levees then confined the ensuing muddy body. Note the levee formation is probably not induced by particle segregation alone since it is also observed for dry granular flows involving spherical equal-size particles (Félix & Thomas, 2004). Figure 1.6(a) shows the lateral levees, which can be used to evaluate the cross-section of the flow, while Fig. 1.6(b) shows a granular levee formed by a debris flow on the alluvial fan. Similar features are also observed for wet-snow avalanches (Jomelli & Bertran, 2001) and pyroclastic flows (Iverson & Vallance, 2001).
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Figure 1.5 Snapshots showing slurry flow discharging from the U.S. Geological Survey Debris-flume Flume and crossing the unconfined, nearly horizontal runout zone. The dark-toned material around the perimeter of the flow was predominantly gravel, while the light-toned material in the center of the flow was liquified mud. Figure reproduced from (Iverson, 2003a); courtesy of Richard M. Iverson.

Figure 1.6 (a) Cross-section of the Malleval stream after a debris flow in August 1999 (Hautes-Alpes, France). (b) Levees left by a debris flow in the Dunant river in July 2006 (Valais, Switzerland); courtesy of Alain Delalune.
1.3.2 Anatomy of powder-snow avalanches

Although there is probably no unique typical outline, powder-snow avalanches are usually made up of two regions when they are in a flowing regime:

- The leading edge is the frontal zone where intense mixing occurs. Motion is produced by the density contrast between the suspension and the surrounding fluid. Since the surrounding fluid is entrained into the current, the snow concentration decreases inside the current, leading, in turn, to a decrease in the buoyancy force unless the current is supplied by a sufficiently strong input of particles. Bed erosion and entrainment of the surrounding fluid into the head are therefore the two main processes that control the bulk dynamics. As long as this balance is maintained, the interface with the ambient fluid is a sharp surface that clearly delineates the avalanche and the ambient air. When air entrainment becomes the prevailing mechanism, the interface becomes a blurred layer. Turbulence is needed in the leading edge to counteract the particle settling: with sufficient turbulence, snow particles (ranging from snow flakes to snow balls) are maintained in suspension. The key condition for the formation and development of a powder cloud is that the vertical velocity fluctuations exceed the particle settling velocity, a condition that is reminiscent of the ‘ignition’ of a gravity current on the ocean floor (Parker, 1982).

- The tail or ‘turbulent wake’ is the volume of fluid behind the head. It is often separated from the leading edge by a billow. The density contrast with the ambient fluid is usually much less marked than for the head. For some events, the powder-snow avalanche leaves behind a motionless cloud, whose size may still be growing as a result of turbulent diffusion; it rapidly settles as turbulent energy falls off.

In the release and runout phase, the structure is usually very different. Indeed, in the release phase, the cloud is not formed (the avalanche looks like a flowing avalanche), while in the runout phase, the cloud collapses and settles to form a vast and thin deposit (thickness lower than 1 m); for many events, it has been observed that the cloud separates from the dense core, which comes to a halt as soon as the slope gradient is too low (typically lower than 20 to 25%). This ‘decoupling’ process is similar in many respects to the abrupt transition observed by Hallworth et al. (1998b) in their laboratory experiments on the instantaneous release of particle-driven gravity currents in a water-filled flume; it probably results from enhanced friction between particles, thus higher dissipation rates in the core than in the dilute cloud. Figure 1.7(a) shows a powder snow avalanche in a flowing regime. The 20-m
high mast and the trees on either side of the avalanche path give a scale of the depth of this avalanche; its velocity was close to 60 m/s. Figure 1.7(b) also shows a powder snow avalanche, but in its runout phase. Note that the depth is much higher than the trees; although its velocity was quite high, this cloud did not cause any damage to the forest, which implies that the impact pressure, thus the bulk density were low. Figure 1.7(c) shows that for this avalanche, part of the avalanche mass was concentrated in a dense core, which stopped prior to reaching the valley bottom.

Figure 1.7 (a) powder snow avalanche in a flowing regime. Photograph taken in the Vallée-de-la-Sionne field site (Switzerland) in January 2004; courtesy of François Dufour, SLF. (a) Runup of a cloud of a powder snow avalanche in a runout phase. Photograph taken at le Roux-d’Abrèes, France in January 2004; courtesy of Maurice Chave. (c) Deposit of the dense core for the same avalanche; courtesy of Hervé Wadier.

There are not many field observations of the internal structure of powder snow avalanches (Issler, 2003; Rammer et al., 2007) and much of our current knowledge stems from what we can infer from small-scale experiments in the laboratory, which were conducted with a partial similitude with real flows. Field observations and laboratory experiments reveal four important aspects:
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- **Existence of eddies**: field measurements (based on radar or pressure-sensor measurements) show that the internal velocity is higher than the front velocity and vary cyclically with time, which was interpreted as the hallmark of rotational flows. Experiments of gravity currents in tanks have shown that the leading edge is associated with a pair of vortices, one located at the leading edge and another one at the rear of the head (see Fig. 1.8). In experiments conducted by Simpson (1972), the development of the flow patterns were made visible using a blend of dense fluid and fine aluminum particles: a stretching vortex occupying the tip region was clearly observed at the leading edge and produced an intense roll-up of fine aluminum particles, which makes it possible to visualize the streamlines and the two vortices; in the upper part of the head, a counter-clockwise rotating vortex occurred. Experiments carried out by Ancey (2004) on finite-volume gravity currents moving down a slope also revealed that the particle cloud was composed of two evident eddies: when the surge involving a glass-bead suspension in water moved from left to right, he observed a small vortex ahead of the front, spinning clockwise, and a large counter-clockwise eddy occupying most of the surge volume. Theoretically, this is in line with the seminal paper of McElwaine (2005) who extended Benjamin’s results by considering steady finite-volume currents down a steep slope, which experience resistance from the surrounding fluid. Like Benjamin (1968), he found that the front makes a $\pi/3$ angle with the bottom line. More recently, Ancey et al. (2006, 2007) worked out analytical solutions to the depth-averaged equations and the Euler equations, which represent the flow of non-Boussinesq currents; they also found that the flow must be rotational and that the head is wedge-shaped.

- **Vertical density stratification**: turbulence is often not sufficient to mix the cloud efficiently and maintain a uniform density through the cloud. Instead, a dense layer forms at the bottom and the density decreases quickly upward (Issler, 2003). For many events, it has also been observed that the dilute component of the avalanche flowed faster than the core and eventually detached from it, which leads to think that there was a sharp transition from the dense basal layer to the dilute upper layer. Typically, from impact force measurement against static obstacles, it was inferred that the basal layer was 1 to 3 m thick, with velocity and (instantaneous) impact pressure as high as 30 m/s and 400 kPa. The transition layer is typically 5 m thick, with kinetic pressure in the 50–100 kPa range. In the dilute upper layer, which can be very thick (as large as 100 m), the kinetic pressure drops to a few pascals, but the velocity is quite high, with typical values close to 60–80 m/s.
Snow entrainment: it alters speed and runout distance. The primary mode of entrainment appeared to be frontal ploughing, although entrainment behind the avalanche front was also observed (Gauer & Issler, 2003; Sovilla & Bartelt, 2006). When there is snow entrainment, the front is wedge-shaped. It can present lobes and clefts, more occasionally digitations, which appear and quickly disappear. The flow depth lies in the 10–50 m range and varies little with distance. Theoretical calculations bring to values of angle of π/3, which seems consistent with field observations (McElwaine, 2005). In absence of entrainment, the front becomes vertical, with a typical nose shape. The surface is diffuse and smoother. The flow depth can be as large as 100 m and quickly varies with distance.

Air entrainment: changes in cloud volume result primarily from the entrainment of the surrounding air. Various mixing processes are responsible for the entrainment of an ambient, less dense fluid into a denser current (or cloud). It has been shown for jets, plumes, and currents that (1) different shear instabilities (Kelvin-Helmoltz, Hölmboe, etc.) can occur at the interface between dense and less dense fluids, (2) the rate of growth of these instabilities is controlled by a Richardson number (Turner, 1973; Fernando, 1991), defined here:

\[ Ri = \frac{g' H \cos \theta}{U^2}, \]  
where \( g' \) denotes the reduced gravity \( g' = g \Delta \bar{\rho}/\rho_a \) and \( \Delta \bar{\rho} = \bar{\rho} - \rho_a \) is the buoyant density. Note that the Richardson number is the inverse square of the Froude number used in hydraulics. The Richardson number can be seen as the ratio of the potential energy \( (g \Delta \bar{\rho} H \cos \theta) \) to the kinetic energy \( (\rho_a U^2) \) of a parcel of fluid at the current interface. Usually a smaller \( Ri \) value implies predominance of inertia effects over the restoring action of gravity, thus greater instability and therefore a higher entrainment rate; it is then expected that the entrainment rate is a decreasing function of the Richardson number. Mixing is observed to occur in gravity currents due to the formation of Kelvin-Helmoltz (K-H) billows at the front, which grow in size, are advected upwards and finally collapse behind the head; the lobe-and-cleft instability is also an efficient mechanism of entrainment (Simpson, 1997). Although the details of the mixing mechanisms are very complex, a striking result of recent research is that their overall effects can be described using simple relations with bulk variables (Turner, 1973; Fernando, 1991). For instance, as regards the volume balance equation, the commonest assumption is to state that the volume variations come from the entrainment of the ambient fluid into the cloud and that the
inflow rate is proportional to the exposed surface area and a characteristic velocity $u_e$: $\dot{V} = E_v S u_e$ where $E_v$ is the bulk entrainment coefficient and $u_e = \sqrt{\bar{\rho}/\rho_a U}$ for a non-Boussinesq current.

Figure 1.8 typical structure of the head of a powder-snow avalanche as interpreted from field measurements and laboratory experiments.

1.4 Fluid-mechanics approach to gravity currents

Gravity-driven flows usually take the appearance of more or less viscous fluids flowing down a slope and this observation has prompted the use of fluid-mechanics tools for describing their motion. However, the impediments to a full fluid-mechanics approach are many: a wide range of particle size (often in the $10^{-3} - 1$-m range), composition that may change with time and/or position, ill-known boundary conditions (e.g., erodible basal surface) and initial conditions, time-dependent flows with abrupt changes (e.g., surge front, instabilities along the free surface), etc. All these difficulties pose great challenges in any fluid-mechanics approach for modelling rapid mass movements and have given impetus to extensive research combining laboratory and field experiments, theory, field observation, and numerical simulations.

Avalanches and debris flows can be considered at different spatial scales (see Fig. 1.9). The larger scale, corresponding to the entire flow, leads to the simplest models. The chief parameters include the location of the gravity center and its velocity. Mechanical behavior is mainly reflected by the friction force $F$ exerted by the bottom (ground or snowpack) on the avalanche. The smallest scale, close to the size of snow particles involved in the avalanches, leads to complicated rheological and numerical problems. The flow characteristics (velocity, stress) are computed at any point of the occupied space. Intermediate models have also been developed. They benefit from being less complex than three-dimensional numerical models and yet more accurate.
than simple ones. Such intermediate models are generally obtained by integrating the equations of motion through the flow depth in a way similar to what is done in hydraulics for shallow water equations.

We start our review of these three approaches with a discussion of the flow regimes (see § 1.4.1). We then briefly describe the rheological behavior of natural materials involved in gravity flows in § 1.4.2. In § 1.4.3, we outline the simplest approach: the sliding-block model, which can be used to give some crude estimates of the speed and dynamic features as well as scaling relations between flow variables and input parameters. A more involved approach consists of taking the depth average of the local governing equations (see § 1.4.4), which enables us to derive a set of partial differential equations for the flow depth $h$ and mean velocity $\bar{u}$. In principle, the local governing equations could be integrated numerically, but the numerical cost is very high and the gain in accuracy is spoiled by the poor knowledge of the rheologic properties or the initial/boundary conditions. Here we confine attention to analytical treatments, which involves working out approximate solutions by using asymptotic expansions of the velocity field (see § 1.4.5).

### 1.4.1 Scaling and flow regimes

Here we will examine how different flow regimes can occur depending on the relative strength of inertial, pressure, and viscous contributions in the governing equations. Dimensional analysis helps clarify the notions of inertia-dominated and friction-dominated regimes. In the analytical computations, we will use the shallowness of sheet flows to derive approximate equations. We also assume that materials that consist of solid particles can be treated
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as continuous materials when they flow, with a dynamic viscosity \( \mu \). This is a very crude assumption, but this will aid in setting the scene.

We consider a shallow layer of fluid flowing over a rigid impermeable plane inclined at an angle \( \theta \) (see Fig. 1.10). The fluid is viscoplastic and incompressible; its density is denoted by \( \rho \) and its bulk viscosity by \( \eta = \tau / \dot{\gamma} \). The ratio \( \epsilon = H_\ast / L_\ast \) between the typical vertical and horizontal length scales, \( H_\ast \) and \( L_\ast \) respectively, is assumed to be small. The streamwise and vertical coordinates are denoted by \( x \) and \( y \), respectively.

A two-dimensional flow regime is assumed, namely any cross-stream variation is neglected. The depth of the layer is given by \( h(x, t) \). The horizontal and vertical velocity components of the velocity \( \mathbf{u} \) are denoted by \( u \) and \( v \), respectively. The fluid pressure is referred to as \( p(x, y, t) \), where \( t \) denotes time. The surrounding fluid (assumed to be air) is assumed to be dynamically passive (i.e., inviscid and low density compared to the moving fluid) and surface tension is neglected, which implies that the stress state at the free surface is zero.

The governing equations are given by the mass and momentum balance equations

\[ \nabla \cdot \mathbf{u} = 0, \tag{1.2} \]

\[ \rho \frac{d \mathbf{u}}{dt} = \rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \rho \mathbf{g} - \nabla p + \nabla \cdot \mathbf{\sigma}, \tag{1.3} \]

supplemented by the following boundary conditions at the free surface

\[ v(x, h, t) = \frac{dh}{dt} = \frac{\partial h}{\partial t} + u(x, h, t) \frac{\partial h}{\partial x}, \quad v(x, 0, t) = 0. \tag{1.4} \]
There are many ways of transforming these governing equations into dimensionless expressions (Liu & Mei, 1990a; Balmforth & Craster, 1999; Keller, 2003; Ancey & Cochard, 2009). Here we depart slightly from the presentation given by Liu & Mei (1990a). The characteristic streamwise and vertical velocities, the timescale, the typical pressure, and the order of magnitude of bulk viscosity are referred to as $U_*, V_*, T_*, P_*$, and $\eta_*$, respectively. Moreover, in addition to the lengthscale ratio $\epsilon$, we introduce the following dimensionless numbers that characterize free-surface, gravity-driven flows: the flow Reynolds number and the Froude number

$$\text{Re} = \frac{\rho U_* H_*}{\eta_*} \quad \text{and} \quad \text{Fr} = \frac{U_*}{\sqrt{g H_* \cos \theta}}.$$  

The following dimensionless variables will be used in this section:

$$\hat{u} = \frac{u}{U_*}, \hat{v} = \frac{v}{V_*}, \hat{x} = \frac{x}{L_*}, \hat{y} = \frac{y}{H_*}, \text{ and } \hat{t} = \frac{t}{T_*}.$$  

A natural choice for $T_*$ is $T_* = L_*/U_*$. The stresses are scaled as follows:

$$\hat{\sigma}_{xx} = \frac{\eta_* U_*}{L_*} \sigma_{xx}, \hat{\sigma}_{xy} = \frac{\eta_* U_*}{H_*} \sigma_{xy}, \hat{\sigma}_{yy} = \frac{\eta_* U_*}{L_*} \sigma_{yy}, \text{ and } \hat{p} = \frac{p}{P_*},$$

where $\sigma_{xx}$, $\sigma_{xy}$, and $\sigma_{yy}$ are the normal stress in the $x$ direction, the shear stress, and the normal stress in the $x$ direction, respectively. Here we are interested in free-surface flows. This leads us to set $P_* = \rho g H_* \cos \theta$, since we expect that, to leading order, the pressure adopts a hydrostatic distribution (see below). If we define the vertical velocity scale as $V_* = \epsilon U_*$, the mass balance equation (1.2) takes the following dimensionless form

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0. \quad (1.5)$$

Substituting the dimensionless variables into the momentum balance equation (1.3) leads to

$$\epsilon \text{Re} \frac{d\hat{u}}{dt} = \frac{\epsilon \text{Re}}{\text{Fr}^2} \left( \frac{1}{\epsilon} \tan \theta - \frac{\partial \hat{p}}{\partial \hat{x}} \right) + \epsilon^2 \frac{\partial \hat{\sigma}_{xx}}{\partial \hat{x}} + \frac{\partial \hat{\sigma}_{xy}}{\partial \hat{y}}, \quad (1.6)$$

$$\epsilon^3 \text{Re} \frac{d\hat{v}}{dt} = \frac{\epsilon \text{Re}}{\text{Fr}^2} \left( -1 - \frac{\partial \hat{p}}{\partial \hat{y}} \right) + \epsilon^2 \frac{\partial \hat{\sigma}_{xy}}{\partial \hat{x}} + \epsilon^2 \frac{\partial \hat{\sigma}_{yy}}{\partial \hat{y}}. \quad (1.7)$$

The momentum balance equation expresses a balance between gravity acceleration, inertial terms, pressure gradient, and viscous dissipation, whose order of magnitude is $\rho g \sin \theta$, $\rho U_*^2/L_*$, $P_*/L_*$, and $\eta_* U_*/H_*^2$, respectively. Depending on the values considered for the characteristic scales, different
types of flow regime occur. At least four regimes, where two contributions prevail compared to the others, could be achieved in principle:

1. **Inertial regime**, where inertial and pressure-gradient terms are of the same magnitude. We obtain

   \[ U_\ast = \sqrt{gH_\ast \cos \theta}. \]

   The order of magnitude of the shear stress is \( \partial \sigma_{xy}/\partial y = \rho g O(\epsilon^{-1} \text{Re}^{-1}) \).

   This regime occurs when: \( \epsilon \text{Re} \gg 1 \) and \( \text{Fr} = O(1) \).

2. **Diffusive regime**, where the pressure gradient is balanced by viscous stresses within the bulk. In that case, we have

   \[ U_\ast = \frac{\rho g \cos \theta H_\ast^2}{\eta_\ast L_\ast}. \]

   Inertial terms must be low compared to the pressure gradient and the slope must be shallow \( (\tan \theta \ll \epsilon) \). This imposes the following constraint: \( \epsilon \text{Re} \ll 1 \). We deduced that \( \text{Fr}^2 = O(\epsilon \text{Re}) \ll 1 \).

3. **Visco-inertial regime**, where inertial and viscous contributions are nearly equal. In that case, we have

   \[ U_\ast = \frac{1}{\epsilon} \frac{\eta_\ast}{\rho H_\ast}. \]

   The pressure gradient must be low compared to the viscous stress, which entails the following condition \( \eta_\ast \gg \epsilon \rho \sqrt{gH_\ast^3} \).

   We obtain \( \epsilon \text{Re} \sim 1 \) and \( \text{Fr} = \eta_\ast/(\rho \epsilon \sqrt{gH_\ast^3}) \gg 1 \).

4. **Nearly steady uniform regime**, where the viscous contribution matches gravity acceleration. In that case, we have

   \[ U_\ast = \frac{\rho g \sin \theta H_\ast^2}{\eta_\ast}. \]

   Inertia must be negligible, which means \( \epsilon \ll 1 \) (stretched flows). We obtain \( \text{Re} = O(\text{Fr}^2) \) and \( \tan \theta \gg \epsilon \) (mild slopes).

In the **inertia-dominated regime**, the rheological effects are so low that they can be neglected and the final governing equations are the Euler equations; this approximation can be used to describe high-speed flows such as powder snow avalanches in the flowing regime (Ancey et al., 2007). The **visco-inertial regime** is more spurious and has no specific interest in geophysics, notably because the flows are rapidly unstable. More interesting is the **diffusive regime** that may achieved for very slow flows on gentle slopes \( (\theta \ll 1) \), typically when flows come to rest, or within the head (Liu & Mei, 1990b; Balmforth et al., 2002; Ancey & Cochard, 2009; Ancey et al., 2009).
will further describe this regime in § 1.5.1. The nearly-steady regime will be exemplified in § 1.5.1 within the framework of the kinematic-wave approximation.

Note that the partitioning into four regimes holds for viscous (Newtonian) fluids and non-Newtonian materials for which the bulk viscosity does not vary significantly with shear rate over a sufficiently wide range of shear rates. In the converse case, further dimensionless groups (e.g., the Bingham number \( \text{Bi} = \tau_c H_0 / (\mu U_0) \)) with \( \tau_c \) the yield stress) must be introduced, which makes this classification more complicated.

### 1.4.2 Rheology

In geophysical fluid mechanics, there have been many attempts to describe the rheological behavior of natural materials (Ancey, 2007). However, since rheometric experiments are no way easy (see below), scientists use proxy procedures to characterize the rheological behavior of natural materials. Interpreting the traces of past events (e.g., shape of deposits), running small-scale experiments with materials mimicking the behavior of natural materials, and making analogies with idealized materials are common approaches to this issue. Because of a lack of experimental validation, there are many points of contention within the different communities working on geophysical flows. A typical example is provided by the debate around the most appropriate constitutive equation for describing sediment mixtures mobilized by debris flows (Iverson, 2003a): a certain part of the debris flow community uses soil-mechanics concepts (Coulomb behavior), while another part prefers viscoplastic models. A third category merges the different concepts from soil and fluid mechanics to provide general constitutive equations.

Over the last 20 years, a large number of experiments have been carried out to test the rheological properties of natural materials. The crux of the difficulty lies in the design of specific rheometers compatible with the relatively large size of particles involved in geophysical flows. Coaxial-cylinder (Couette) rheometers and inclined flumes are the most popular geometries. Another source of trouble stems from disturbing effects such as particle migration and segregation, flow heterogeneities, fracture, layering, etc. These effects are often very pronounced with natural materials, which may explain the poor reproducibility of rheometric investigations (Major & Pierson, 1992; Contreras & Davies, 2000; Iverson, 2003b). Poor reproducibility, complexity in the material response, and data scattering have at times been interpreted as the failure of the one-phase approximation for describing rheological properties (Iverson, 2003b). In fact, these experimental problems demonstrate
above all that the bulk behavior of natural material is characterized by fluctuations that can be as wide as the mean values. As for turbulence and Brownian motion, we should describe not only the mean behavior, but also the fluctuating behavior to properly characterize the rheological properties. For concentrated colloidal or granular materials (Lootens et al., 2003; Tsai et al., 2001), experiments on well-controlled materials have provided evidence that to some extent, these fluctuations originate from jamming in the particle network (creation of force vaults sustaining normal stress and resisting against shear stress, both of which suddenly relax). Other processes such as ordering, aging, and chemical alteration occur in natural slurries, which may explain their time-dependent properties (Marquez et al., 2006). Finally, there are disturbing effects (e.g., slipping along the smooth surfaces of a rheometer), which may bias measurement.

Snow is a very special material. To illustrate the diversity of materials involved in snow avalanches, Fig. 1.11 reports different types of snow observed in avalanche deposits. Experiments have been done in the laboratory to characterize its rheological behavior. Authors such as Dent & Lang (1982) and Maeno (1993) have measured the velocity profile within snow flows and generally deduced that snow generates a non-Newtonian viscoplastic flow, whose properties depend a great deal on density. Carrying these laboratory results over real avalanches is not clearly reliable due to size-scale effects and similarity conditions. Furthermore, given the severe difficulties inherent to snow rheometry (sample fracture during shearing tests, variation in the snow microstructure resulting from thermodynamic transformations of crystals, etc.), properly identifying the constitutive equation of snow with modern rheometers is out of reach for the moment. More recently, Ancey & Meunier (2004) showed how avalanche-velocity records can be used to determine the bulk frictional force; a striking result is that the bulk behavior of most snow avalanches can be approximated using a Coulomb frictional model. Kern et al. (2004) and Kern et al. (2009) ran outdoor and field experiments to measure shear-rate profiles inside snow flows to infer rheological properties; this is rather encouraging and clears the way for precise rheometrical investigations of real snow avalanches.

Since there are little sound field or laboratory data available on the basic rheological processes involved in avalanche release and flow, all avalanche-dynamics models proposed so far rely on analogy with other physical phenomena: typical examples include analogies with granular flows (Savage & Hutter, 1989; Savage, 1989; Tai et al., 2001; Cui et al., 2007), Newtonian fluids (Hunt, 1994), power-law fluids (Norem et al., 1986), and viscoplastic flows (Dent & Lang, 1982; Ancey, 2007). From a purely rheological point of
view, these models rely on a purely speculative foundation. Indeed, most of the time, the rheological parameters used in these models have been estimated by matching the model predictions (such as the leading-edge velocity and the run-out distance) with field data (Buser & Frutiger, 1980; Dent & Lang, 1980; Ancey et al., 2004). However, this obviously does not provide evidence that the constitutive equation is appropriate.

Figure 1.11 Different types of snow observed in avalanche deposits. (a) Block of wet snow (size: 1 m). (b) Slurry of dry snow including weak snow-balls formed during the course of the avalanche (the heap height was approximately 2 m). (c) Ice balls involved in a huge avalanche coming from the North face of the Mont Blanc (France); the typical diameter was 10 cm. (d) Sintered snow forming broken slabs (typical length: 40 cm, typical thickness 10 cm).

For debris flows, natural suspensions are made up of a great diversity of grains and fluids. This observation motivates fundamental questions: how to distinguish between the solid and fluid phases? What is the effect of colloidal particles in a suspension composed of coarse and fine particles? When the particle size distribution is bimodal (i.e., we can distinguish between fine and coarse particles), the fine fraction and the interstitial fluid form a viscoplastic fluid embedding the coarse particles, as suggested by Sengun & Probstein
Gravity flow on steep slope

(1989); this leads to a wide range of viscoplastic constitutive equations, the most common being the Herschel-Bulkley model. The bimodal-suspension approximation usually breaks for poorly sorted slurries. In that case, following Iverson and his co-workers (Iverson, 1997, 2005), Coulomb plasticity can help understand the complex, time-dependent rheological behavior of slurries.

When the bulk is made up of fine colloidal particles, phenomenological laws are used to describe rheological behavior. One of the most popular is the Herschel-Bulkley model, which generalizes the Bingham law

\[ \tau = \tau_c + K \dot{\gamma}^n, \] (1.8)

with \( \tau_c \) the yield stress, \( K \) and \( n \) two constitutive parameters. In practice, this phenomenological expression successfully describes the rheological behavior of many materials over a sufficiently wide range of shear rates, except at very low shear rates.

When the bulk is made up of coarse noncolloidal particles, Coulomb friction at the particle level imparts its key properties to the bulk, which explains (i) the linear relationship between the shear stress \( \tau \) and the effective normal stress \( \sigma' = \sigma - p \) (with \( p \) the interstitial pore pressure)

\[ \tau = \sigma' \tan \varphi, \] (1.9)

and (ii) the non-dependence of the shear stress on the shear rate \( \dot{\gamma} \). Some authors have suggested that in high-velocity flows, particles undergo collisions, which gives rise to a regime referred to as the frictional-collisional regime. The first proposition of bulk stress tensor seems to be attributable to Savage (1982), who split the shear stress into frictional and collisional contributions

\[ \tau = \sigma \tan \varphi + \mu(T)\dot{\gamma}, \] (1.10)

with \( T \) the granular temperature. Elaborating on this model, Ancey & Evesque (2000) suggested that there is a coupling between frictional and collisional processes. Using heuristic arguments on energy balance, they concluded that the collisional viscosity should depend on the Coulomb number \( \text{Co} = \rho_p a^2 \dot{\gamma}^2 / \sigma \) (called the inertial number by Jop et al. (2005) and part of the French granular-flow community) to allow for this coupling in a simple way

\[ \tau = \sigma \tan \varphi + \mu(\text{Co})\dot{\gamma}. \] (1.11)

Jop et al. (2005) proposed a slightly different version of this model, where
both the bulk frictional and collisional contributions collapse into a single term, which is a function of the Coulomb number

$$\tau = \sigma \tan \varphi(\text{Co}).$$

(1.12)

Contrasting with other propositions, (Josserand et al., 2004) stated that the key variable in shear stress was the solid concentration $\phi$ rather than the Coulomb number

$$\tau = K(\phi)\sigma + \mu(\phi)\dot{\gamma}^2,$$

(1.13)

with $K$ a friction coefficient. Every model is successful in predicting experimental observations for some flow conditions, but to date, none is able to describe the frictional-collisional regime for a wide range of flow conditions and material properties.

### 1.4.3 Sliding-block and box models

The simplest model for computing the propagation speed of a gravity current proceeds by assuming that there is no downstream variation in flow properties (i.e., density, friction) within the flowing bulk. Several classes of models have been developed:

- **Sliding block model:** the flow is assumed to behave as a rigid block experiencing a frictional force. The early models date back to the beginning of the 20th century (Mougin, 1922). The method was then extended by Voellmy (1955), who popularized it. Many models have elaborated on Voellmy’s work. The Voellmy–Salm–Gubler (VSG) model and the Perla–Cheng–McClung model are probably the best-known avalanche-dynamics models used throughout the world (Salm et al., 1990; Perla et al., 1980).

- **Box model:** the model relaxes the rigidity assumption of the sliding-block model by considering that the current behaves as a deformable rectangular box of length $\ell$ and height $h$. Mass conservation implies that the volume of this rectangle is known. For inertia-dominated flows, the Froude number at the leading edge is usually given by a boundary condition such as the von kármán condition: $Fr = u_f / \sqrt{g' h} = cste$ (with $g'$ the reduced gravity acceleration, see (1.1), and $u_f$ the front velocity). Since box models have been developed for flows on horizontal surface, they are not well suited to studying flows on steep slope.

- **Cloud model:** the current is assumed to behave as a deformable body, whose shape keeps the same aspect. The governing equations are given by the mass and momentum conservation equations for a mass-varying body.
Kulikovskiy & Svehnikova (1977) presented a theoretical model (the KS model), in which the cloud was defined as a semielliptic body, whose volume varied with time. The cloud density can vary depending on air and snow entrainment. Kulikovskiy & Svehnikova (1977) obtained a set of four equations describing the mass, volume, momentum, and Lagrangian kinetic energy balances. Extensions to the KS model were then added to account for the influence of bottom drag, three-dimensional spreading, etc.

Simple models have been developed for almost 80 years in order to crude estimations of avalanche features (velocity, pressure, runout distance). They are used extensively in engineering throughout the world. Despite their simplicity and approximate character, they can provide valuable results, the more so as their parameters and the computation procedures combining expert rules and scientific basis have benefited from many improvements over the last few decades (Ancey et al., 2003; Ancey & Meunier, 2004; Ancey et al., 2004; Ancey, 2005).

1.4.4 Depth-averaged equations

The most common method for solving free-surface problems is to take the depth-average of the local equations of motion. In the literature, this method is referred to as the Saint-Venant approach since it was originally developed to compute floods in rivers.

We consider flows without entrainment of the surrounding fluid and variation in density: $\rho \approx \bar{\rho}$ (see §1.6.2 for flow with entrainment). Accordingly the bulk density may be merely replaced by its mean value. In this context, the equations of motion may be inferred in a way similar to the usual procedure used in hydraulics to derive the shallow water equations (or Saint-Venant equations): it involves integrating the momentum and mass balance equations over the depth. As such a method has been extensively used in hydraulics for water flow (Chow, 1959) as well for non-Newtonian fluids (Savage & Hutter, 1991; Bouchut et al., 2003); we briefly recall the principle and then directly provide the resulting equations of motion. Let us consider the local mass balance: $\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{u}) = 0$. Integrating this equation over the flow depth leads to:

$$\int_0^{h(x,t)} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dy = \frac{\partial}{\partial x} \int_0^h u(x, y, t) dy - u(h) \frac{\partial h}{\partial x} - v(x, h, t) - v(x, 0, t),$$

\hspace{1cm} (1.14)
1.4 Fluid-mechanics approach to gravity currents

where \( u \) and \( v \) denote the \( x \)- and \( y \)-component of the local velocity. At the free surface and the bottom, the \( y \)-component of velocity satisfies the following boundary conditions:

\[
v(x, h, t) = \frac{dh}{dt} = \frac{\partial h}{\partial t} + u(x, h, t) \frac{\partial h}{\partial x},
\]

\[v(x, 0, t) = 0.\]  

(1.15)  

(1.16)

We easily deduce:

\[
\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} = 0,
\]

(1.17)

where we have introduced depth-averaged values defined as:

\[
\bar{f}(x, t) = \frac{1}{h(x, t)} \int_0^{h(x, t)} f(x, y, t) dy.
\]

(1.18)

The same procedure is applied to the momentum balance equation: \( \rho \frac{d\bar{u}}{dt} = \rho g + \nabla \cdot \sigma \), where \( \sigma \) denotes the stress tensor. Without difficulty, we can deduce the averaged momentum equation from the \( x \)-component of the momentum equation:

\[
\bar{\rho} \left( \frac{\partial \bar{h} \bar{u}}{\partial t} + \frac{\partial \bar{h} \bar{u}^2}{\partial x} \right) = \bar{\rho} g h \sin \theta + \frac{\partial h \bar{\sigma}_{xx}}{\partial x} - \tau_p,
\]

where we have introduced the bottom shear stress: \( \tau_p = \sigma_{xy}(x, 0, t) \). In the present form, the motion equation system (1.17)–(1.19) is not closed since the number of variables exceeds the number of equations. A common approximation involves introducing a parameter (sometimes called the Boussinesq momentum coefficient) which links the mean velocity to the mean square velocity:

\[
\bar{u}^2 = \frac{1}{h} \int_0^h u^2(y) dy = \alpha \bar{u}^2.
\]

(1.20)

Usually \( \alpha \) is set to unity, but this may cause trouble when computing the head structure (Hogg & Pritchard, 2004; Ancey et al., 2006, 2007). A point often neglected is that the shallow-flow approximation is in principle valid for flow regimes that are not too far away from a steady uniform regime. In flow parts where there are significant variations in the flow depth (e.g. at the leading edge and when the flow widens or narrows substantially), corrections should be made to the first-order approximation of stress. Recent studies however showed that errors made with the shallow-flow approximation for
the leading edge are, however, not significant (Ancey et al., 2007; Ancey & Cochard, 2009; Ancey et al., 2009).

A considerable body of work has been published on this method for Newtonian and non-Newtonian fluids, including viscoplastic (Coussot, 1997; Huang & García, 1998; Siviglia & Cantelli, 2005), power-law (Fernández-Nieto et al., 2010), and granular materials (Savage & Hutter, 1989; Gray et al., 1998; Pouliquen & Forterre, 2002; Iverson & Denlinger, 2001; Bouchut et al., 2003; Chugunov et al., 2003; Pudasaini & Hutter, 2003; Kerswell, 2005).

With depth-averaged flow models, the limitations of simple models are alleviated. For instance it is possible to compute the spreading of avalanches in their runout zone or relate mechanical parameters used in the models to the rheological properties of snow. As far as we know, the early depth-averaged models were developed in the 1970s by Russian scientists (Bozhinskiy & Losev, 1998) and French researchers (Brugnot & Pochat, 1981; Vila, 1986) for flowing avalanches. For airborne avalanches, the first stage was probably the model developed by Parker et al. (1986), which, though devoted to submarine turbidity currents, contains almost all the ingredients used in subsequent models of airborne avalanches. Considerable progress in the development of numerical depth-averaged models has been made possible thanks to the increase in computer power and breakthrough in the numerical treatment of hyperbolic partial differential equation systems (LeVeque, 2002).

1.4.5 Asymptotic expansions

On many occasions, flows are not equilibrium, but slightly deviates from it. In this context, it is often convenient to use asymptotic expansions for the velocity field:

\[ u(x, y, t) = u_0(x, y, t) + \epsilon u_1(x, y, t) + \epsilon^2 u_2(x, y, t) + \cdots , \]

where \( \epsilon \) is a small number (e.g., the aspect ratio \( \epsilon = H_s/L_s \) in Eqs. (1.6)–(1.7)) and \( u_i(x, y, t) \) are functions to be determined; usually, \( u_0 \) is the velocity field when the flow is at equilibrium and \( u_i \) represents perturbations to this equilibrium state. Substituting \( u \) by this expansion into the local governing equations such as Eqs. (1.6)–(1.7) leads to a hierarchy of equations of increasing order. Most of the time, only the zero-order solution and the first-order correction are computed. Examples will be provided with the computation of a viscoplastic sheet flow in § 1.5.1.
1.5 Dense flows

We address the issue of dense flows, for which the effect of the surrounding air is neglected. We first illustrate the sliding-block approach by outlining the Voellmy-Salm-Gubler model, which is one of the most popular models worldwide for computing the main features of extreme snow avalanches (Salm et al., 1990). In § 1.5.1, we see two applications of the flow-depth averaged equations (frictional and viscoplastic fluids). We end this section with the use of asymptotic expansions to describe the motion of viscoplastic sheet flows (see § 1.5.1).

1.5.1 Simple models

The avalanche is assumed to behave as a rigid body, which moves along an inclined plane. The position of the center of mass is given by its abscissa $x$ in the downward direction. The momentum equation is

$$\frac{du}{dt} = g \sin \theta - \frac{F}{m},$$

(1.21)

with $m$ the avalanche mass, $u$ its velocity, $\theta$ the mean slope of the path, and $F$ the frictional force. In this model, the sliding block is subject to a frictional force combining a solid-friction component and a square-velocity component:

$$F = mg \frac{u^2}{\xi h} + \mu mg \cos \theta,$$

(1.22)

with $h$ the mean flow depth of the avalanche, $\mu$ a friction coefficient related to the snow fluidity, and $\xi$ a coefficient of dynamic friction related to path roughness. If these last two parameters cannot be measured directly, they can be adjusted from several series of past events. It is generally accepted that the friction coefficient $\mu$ only depends on the avalanche size and ranges from 0.4 (small avalanches) to 0.155 (very large avalanches) (Salm et al., 1990); in practice, lower values can be observed for large-volume avalanches (Ancey et al., 2004). Likewise, the dynamic parameter $\xi$ reflects the influence of the path on avalanche motion. When an avalanche runs down a wide open rough slope, $\xi$ is close to 1000. Conversely, for avalanches moving down confined straight gullies, $\xi$ can be taken as being equal to 400. In a steady state, the velocity is directly inferred from the momentum balance equation:

$$u = \sqrt{\xi h \cos \theta (\tan \theta - \mu)}.$$

(1.23)

According to this equation two flow regimes can occur depending on path inclination. For $\tan \theta > \mu$, (1.23) has a real solution and a steady regime
can occur. For \( \tan \theta < \mu \), there is no real solution: the frictional force (1.22) outweighs the downward component of the gravitational force. It is therefore considered that the flow slows down. The point of the path for which \( \tan \theta = \mu \) is called the characteristic point (point \( P \)). It plays an important role in avalanche dynamics since it separates flowing and runout phases. In the stopping area, we deduce from the momentum equation that the velocity decreases as follows:

\[
\frac{1}{2} \frac{du^2}{dx} + u^2 \frac{g}{\xi h} = g \cos \theta (\tan \theta - \mu) .
\]  \hspace{1cm} (1.24)

The runout distance is easily inferred from (1.24) by assuming that at a point \( x = 0 \), the avalanche velocity is \( u_p \). In practice the origin point is point \( P \) but attention must be paid in the fact that, according to (1.23), the velocity at point \( P \) should be vanishing; a specific procedure has been developed to avoid this shortcoming (Salm et al., 1990). Neglecting the slope variations in the stopping zone, we find:

\[
x_a = \frac{\xi h}{2g} \ln \left( 1 + \frac{u_p^2}{\xi h \cos \theta (\mu - \tan \theta)} \right) .
\]  \hspace{1cm} (1.25)

This model enables us to easily compute the runout distance, the maximum velocities reached by the avalanche on various segments of the path, the flow depth (by assuming that the mass flow rate is constant and given by the initial flow rate just after the release), and the impact pressure.

**Depth-averaged equations**

The Saint-Venant equations consist of the following depth-averaged mass and momentum balance equations

\[
\frac{\partial h}{\partial t} + \frac{\partial \bar{h} \bar{u}}{\partial x} = 0 ,
\]  \hspace{1cm} (1.26)

\[
\bar{\rho} \left( \frac{\partial \bar{h} \bar{u}}{\partial t} + \frac{\partial \bar{h} \bar{u}^2}{\partial x} \right) = \bar{\rho} g h \sin \theta - \frac{\partial \bar{h} \bar{p}}{\partial x} + \frac{\partial h \bar{\sigma}_{xx}}{\partial x} - \tau_b ,
\]  \hspace{1cm} (1.27)

where we have introduced the bottom shear stress \( \tau_b = \sigma_{xy}(x,0,t) \) and we assume \( \bar{u}^2 = \bar{\bar{u}}^2 \); the flow-depth averaged pressure is found to be lithostatic

\[
\bar{\rho} = \frac{1}{2} \bar{\rho} g h \cos \theta .
\]

Within the framework of the long-wave approximation, we assume that longitudinal motion outweighs vertical motion: for any quantity \( m \) related to motion, we have \( \partial m / \partial y \gg \partial m / \partial x \). This allows us to consider that every
vertical slice of flow can be treated as if it was locally uniform. In such conditions, it is possible to infer the bottom shear stress by extrapolating its steady-state value and expressing it as a function of \( \bar{u} \) and \( h \). For instance, for viscoplastic fluids, a common constitutive equation is the Herschel-Bulkley law (1.8). By relating the bottom shear rate to the flow-depth averaged velocity, Coussot (1997) obtained the following bottom shear stress

\[
\tau_b = \mu \left( \frac{1 + 2n}{1 + n} \right)^n \frac{\bar{u}^n}{h_0^{n+1}((2 + n^{-1})h - h_0)^n},
\]

for Herschel-Bulkley fluids.

For Coulomb materials, the same procedure can be repeated. The only modification concerns the momentum balance equation (1.27), which takes the form (Savage & Hutter, 1989; Iverson & Denlinger, 2001)

\[
\bar{\rho} \left( \frac{\partial \bar{h} \bar{u}}{\partial t} + \frac{\partial \bar{h} \bar{u}^2}{\partial x} \right) = \bar{\rho} g h \left( \sin \theta - k \cos \theta \frac{\partial h}{\partial x} \right) - \tau_b,
\]

with \( k \) a proportionality coefficient between the normal stresses \( \sigma_{xx} \) and \( \sigma_{yy} \), which is computed by assuming limiting Coulomb equilibrium in compression (\( \partial_x \bar{u} < 0 \)) or extension (\( \partial_x \bar{u} > 0 \)); the coefficient is called the active/passive pressure coefficient. In Eq. (1.28), the bottom shear stress can be computed by using the Coulomb law \( \tau_b = (\sigma_{yy}|_{y=0} - p_b) \tan \varphi \), with \( \sigma_{yy}|_{y=0} = \bar{\rho} gh \cos \theta \) and \( p_b \) the pore pressure at the bed level.

Analytical solutions can be obtained for the Saint-Venant equations. Most of them were derived by seeking self-similarity solutions; see (Savage & Noguchi, 1988; Savage & Hutter, 1989; Chugunov et al., 2003) for the Coulomb model and (Hogg & Pritchard, 2004) for viscoplastic and hydraulic models. Some solutions can also be obtained using the method of characteristics. We are going to see two applications based on these methods.

In the first application, we use the fact that the Saint-Venant equations for Coulomb materials are structurally similar to those used in hydraulics when the bottom drag can be neglected. The only difference lies in the nonhydrostatic pressure term and the source term (bottom shear stress). However, using a change in variable makes it possible to retrieve the usual shallow-water equations and seek similarity solutions to derive the Ritter solutions (Mangeney et al., 2000; Karelsky et al., 2000; Kerswell, 2005). The Ritter solutions are the solutions to the so-called dam-break problem, where an infinite volume of material at rest is suddenly released and spreads over a dry bed (i.e., no material laying along the bed). Much attention has been paid to this problem, notably in geophysics because it is used as a paradigm.
for studying rapid surge motion. We pose

\[ x^* = x - \frac{\delta t^2}{2}, \quad t^* = t, \quad u^* = u - \delta t, \quad \text{and} \quad h^* = h, \]

where we introduced the parameter \( \delta = g \cos \theta (\tan \theta - \mu) \). We deduce

\[ \frac{\partial h^*}{\partial t^*} + \frac{\partial h^* u^*}{\partial x^*} = 0, \quad (1.29) \]

\[ \frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + gk \cos \theta \frac{\partial h^*}{\partial x^*} = 0. \quad (1.30) \]

For the dam-break problem, the initial and boundary conditions are

\[ -\infty < x < \infty, \quad u(x, 0) = 0, \]

\[ x < 0, \quad h(x, 0) = h_i, \quad (1.31) \]

\[ x > 0, \quad h(x, 0) = 0. \]

The analytical solutions to Eqs. (1.29–1.30) are the well-known Ritter solutions. We are looking for a similarity solution in the form (Gratton & Vigo, 1994)

\[ \bar{u}^* = t^{\beta/\alpha} U(\zeta^*), \quad h^* = t^{\gamma/\alpha} H(\zeta^*), \]

with \( \zeta^* = x^*/t^\alpha \) the similarity variable, and \( H \) and \( U \) two unknown functions. Substituting \( \bar{u}^* \) and \( h^* \) with their similarity forms into (1.29–1.30), we find: \( \beta + \alpha = 1 \) and \( \gamma + 2\alpha = 2 \). For this solution to satisfy the initial and boundary conditions, we must pose \( \beta = \gamma = 0 \), hence \( \alpha = 1 \). We then infer

\[ \left( \begin{array}{cc} H & U - \zeta^* \\ U - \zeta^* & kg \cos \theta \end{array} \right) \cdot \left( \begin{array}{c} U' \\ H' \end{array} \right) = 0, \]

where the prime denotes the \( \zeta^* \)-derivative. For this system to admit a non-constant solution, its determinant must vanish, which leads to \( kgH \cos \theta = (U - \zeta^*)^2 \). On substituting this relation into the system above, we deduce \( U' = 2\zeta^*/3 \), thus \( U = 2(\zeta^* + c)/3 \), where \( c \) is a constant of integration, \( H = 4(c - \frac{1}{2}\zeta^*)^2/(9kg \cos \theta) \). The constant \( c \) is found using the boundary conditions and by assuming that the undisturbed flow slides at constant velocity \( \delta t \): \( c = \sqrt{kg h_i \cos \theta} \). Returning to the original variables, we find

\[ \bar{u}(x, \ t) = \bar{u}^* + \delta t = \frac{2}{3} \left( \frac{x}{t} + \delta t + c \right), \quad (1.32) \]

\[ h(x, \ t) = \frac{1}{9kg \cos \theta} \left( -\frac{x}{t} + \frac{\delta}{2}t + 2c \right)^2. \quad (1.33) \]
The boundary conditions also imply that the solution is valid over the \( \zeta \)-range \([-c - \delta t, 2c + \delta t/2]\); the lower bound corresponds to the upstream condition \( \bar{u} = 0 \), while the upper bound is given by the downstream condition \( h = 0 \). It is worth noting that the front velocity \( u_f = 2c + \delta t/2 \) is constantly increasing or decreasing depending on the sign of \( \delta \). When \( \delta < 0 \) (friction in excess of slope angle), the front velocity vanishes at \( t = 4c/|\delta| \). Figure 1.12 shows that the shape of the tip region is parabolic at short times (\( \delta t \ll c \)), in agreement with experimental data (Balmforth & Kerswell, 2005; Siavoshi & Kudrolli, 2005). Solutions corresponding to finite released volumes were also obtained by Ancey et al. (2008), Hogg (2006), and Savage & Nohguchi (1988); Savage & Hutter (1989).

In the second application, we use the method of characteristics to find a solution to the governing equations for Bingham flows that are stretched thin layers when they are nearly steady uniform. For mild slopes, when the aspect ratio \( \epsilon \) is very low, the inertial and pressure contributions can be neglected (see dimensional analysis above). This means that the flow-depth averaged velocity is very close to the mean velocity reached for steady uniform flows

\[
\bar{u}_s = u_p \left( 1 - \frac{h_0}{3h} \right),
\]

where \( u_p \) is the plug velocity

\[
u_p = \frac{\rho gh_0^2 \sin \theta}{2\mu},
\]

with \( h \) the flow depth and \( h_0 = h - \tau_c/(\rho g \sin \theta) \) the yield-surface elevation;
Gravity flow on steep slope

$h_0$ must be positive or no steady flow occurs. We then use the kinematic-wave approximation introduced by Lighthill & Whitham (1955) to study floods on long rivers; this approximation was then extensively used in hydraulic applications (Hunt, 1994; Huang & Garcia, 1997, 1998). It involves substituting the mean velocity into the mass balance equation (1.26) by its steady-state value $\bar{u}_s$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} u_p \left( h - \frac{h_0}{3} \right) = 0. \tag{1.34}$$

Introducing the plug thickness $h_p = h - h_0 = \tau_c/(\rho g \sin \theta)$, we obtain an expression that is a function of $h$ and its time and space derivative

$$\frac{\partial h}{\partial t} + K \left( h^2 - hh_p \right) \frac{\partial h}{\partial x} = 0,$$

with $K = \rho g \sin \theta / \mu$. The governing equation takes the form of a nonlinear advection equation, which can be solved using the method of characteristics (LeVeque, 2002).

Using the chain rule for interpreting this partial differential equation (1.34), we can show that it is equivalent to the following ordinary equation

$$\frac{dh}{dt} = 0, \tag{1.35}$$

along the characteristic curve

$$\frac{dx}{dt} = \lambda(h), \tag{1.36}$$

in the $(x, t)$ plane, with $\lambda(h) = Kh \left( h - h_p \right)$. Equation (1.35) shows that the flow depth is constant along the characteristic curve, hence the characteristic curves are straight lines, the slope of which are given by the right-hand side term $\lambda(h)$ in Eq. (1.36). These characteristic curves can be used to solve an initial value problem, where the initial value of $h$ is known over a given interval: $h = h_i(x_i)$ (at $t = 0$). The value of $h$ along each characteristic curve is the value of $h$ at the initial point $x(0) = x_i$. We can thus write

$$h(x, t) = h_i(x_i) = h_i(x - \lambda(h_i(x_i))t).$$

It is worth noting that because of the nonlinearity of Eq. (1.34), a smooth initial condition can generate a discontinuous solution (shock) if the characteristic curves intersect, since at the point of intersection $h$ takes (at least) two values (LeVeque, 2002).
1.5 Dense flows

Sheet flows

Slow motion of a viscoplastic material has been investigated by Liu & Mei (1990a, b), Mei et al. (2001), Coussot et al. (1996), Balmforth & Craster (1999); Balmforth et al. (2002), Matson & Hogg (2007), Ancey & Cochard (2009), and Hogg & Matson (2009).

Here we consider that the shear stress is given by (1.8) with \( n = 1 \). Taking the two dominant contributions in Eqs. (1.6–1.7) and returning to the physical variables, we deduce

\[
\sigma_{xy} = \rho g \cos \theta (h - y) \left( \tan \theta - \frac{\partial h}{\partial x} \right),
\]

\( (1.37) \)

\[
p = \rho g (h - y) \cos \theta.
\]

(1.38)

The bottom shear stress is then found to be \( \tau_b = \sigma_{xy}{\mid}_{y=0} \). For bottom shear stresses in excess of the yield stress \( \tau_c \), flow is possible. When this condition is satisfied, there is a yield surface at depth \( y = h_0 \) within the bulk, along which the shear stress matches the yield stress

\[
\sigma_{xy}{\mid}_{y=h_0} = \rho g \cos \theta (h - h_0) \left( \tan \theta - \frac{\partial h}{\partial x} \right) = \tau_c.
\]

(1.39)

The yield surface separates the flow into two layers (Liu & Mei, 1990a; Balmforth & Craster, 1999): the bottom layer, which is sheared, and the upper layer or plug layer, where the shear rate is nearly zero. Indeed, using an asymptotic analysis, Balmforth & Craster (1999) demonstrated that in the so-called plug layer, the shear rate is close to zero, but nonzero. This result may seem anecdotic, but it is in fact of great importance since it resolves a number of paradoxes raised about viscoplastic solutions.

On integrating the shear-stress distribution, we can derive a governing equation for the flow depth \( h(x, t) \). For this purpose, we must specify the constitutive equation. For the sake of simplicity, we consider a Bingham fluid in one-dimensional flows as Liu & Mei (1990a) did; the extension to Herschel-Bulkley and/or two-dimensional flows can be found in (Balmforth & Craster, 1999; Balmforth et al., 2002; Mei & Yuhi, 2001; Ancey & Cochard, 2009). In the sheared zone, the velocity profile is parabolic

\[
u(y) = \frac{\rho g \cos \theta}{\mu} \left( \tan \theta - \frac{\partial h}{\partial x} \right) \left( h_0 y - \frac{1}{2} y^2 \right) \quad \text{for} \quad y \leq h_0,
\]

while the velocity is constant to leading order within the plug

\[
u(y) = u_0 = \frac{\rho g h_0^2 \cos \theta}{\mu} \left( \tan \theta - \frac{\partial h}{\partial x} \right) \quad \text{for} \quad y \geq h_0,
\]
Gravity flow on steep slope

The flow rate is then

$$q = \int_0^h u(y) \, dy = \frac{\rho g h_0^2 (3h - h_0) \cos \theta}{6 \mu} \left( \tan \theta - \frac{\partial h}{\partial x} \right).$$

(1.40)

Integrating the mass balance equation over the flow depth provides

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0.$$  

(1.41)

Substituting $q$ with its expression (1.40) and the yield surface elevation $h_0$ with Eq. (1.39) into Eq. (1.41), we obtain a governing equation for $h$, which takes the form of a nonlinear diffusion equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left[ F(h, h_0) \left( \frac{\partial h}{\partial x} - \tan \theta \right) \right],$$

(1.42)

with $F = \frac{\rho g h_0^2 (3h - h_0) \cos \theta}{6 \mu}$.

A typical application of this analysis is the derivation of the shape of a viscoplastic deposit. Contrary to a Newtonian fluid, the flow depth of a viscoplastic fluid cannot decrease indefinitely when the fluid spreads out along an infinite plane. Because of the finite yield stress, when it comes to rest, the fluid exhibits a nonuniform flow-depth profile, where the pressure gradient is exactly balanced by the yield stress. On an infinite horizontal plane, the bottom shear stress must equal the yield stress. Using Eq. (1.37) with $\theta = 0$ and $y = 0$, we eventually obtain (Liu & Mei, 1990)

$$\sigma_{xy} |_{y=0} = \tau_c = -\rho g h \frac{\partial h}{\partial x},$$

(1.43)

which, on integrating, provides

$$h(x) - h_i = \sqrt{\frac{2 \tau_c (x_i - x)}{\rho g}},$$

(1.44)

where $h = h_i$ at $x = x_i$ is a boundary condition. This equation shows that the deposit-thickness profile depends on the square root of the distance. This is good agreement with field observations (Coussot et al., 1996); Fig. 1.13 shows the lobe of debris-flow deposit, whose profile can be closely approximated by (1.44).

When the slope is nonzero, an implicit solution for $h(x)$ to Eq. (1.37) is found (Liu & Mei, 1990)

$$\tan \theta (h(x) - h_i) + \frac{\tau_c}{\rho g \cos \theta} \log \left[ \frac{\tau_c - \rho g h \sin \theta}{\tau_c - \rho g h_i \sin \theta} \right] = \tan^2 \theta (x - x_i).$$

(1.45)

The shape of a static two-dimensional pile of viscoplastic fluid was investigated by Coussot et al. (1996), Mei & Yuhi (2001), Osmond & Griffiths
1.6 Dilute inertia-dominated flows

1.6.1 Sliding block model

The first-generation models of airborne avalanches used the analogy of density currents along inclined surfaces. Extending a model proposed by Ellison & Turner (1959) on the motion of an inclined plume, Hopfinger & Tochon-Danguy (1977) inferred the mean velocity of a steady current, assumed to represent the avalanche body behind the head. They found that the front velocity of the current was fairly independent of the bed slope. The sec-

(2001), and Balmforth et al. (2002); the latter derived an exact solution, while the former authors used numerical methods or ad hoc approximations to solve the two-dimensional equivalent to Eq. (1.37). Similarity solutions to Eq. (1.42) have also been provided by Balmforth et al. (2002) in the case of a viscoplastic flow down a gently inclined, unconfined surface with a time-varying source at the inlet. Ancey & Cochard (2009) used matched-asymptotic expansions to build approximate analytical solutions for the movement of a finite volume of Herschel-Bulkley fluid down a flume. Matson & Hogg (2007) and Hogg & Matson (2009) investigated the slumping motion of a fixed volume on a plane or down an inclined slope.

Figure 1.13 Lobes of a debris-flow deposit near the Rif Paulin stream (Hautes-Alpes, France).
ond generation of models has considered the avalanche as a finite-volume turbulent flow of a snow suspension. Kulikovskiy & Sveshnikova (1977) set forth a fairly simple theoretical model (the KS model), in which the cloud was assimilated to a semi-elliptic body whose volume varied with time. The kinematics was entirely described by the mass center position and two geometric parameters of the cloud (the two semi-axes of the ellipsis). The cloud density can vary depending on air and snow entrainments. Kulikovskiy and Sveshnikova obtained a set of four equations describing the mass, volume, momentum, and Lagrangian kinetic energy balances. The idea was subsequently redeveloped by many authors including Beghin et al. (1981), Beghin & Brugnot (1983), Fukushima & Parker (1990), Beghin & Olagne (1991), Fukushima et al. (2000), Ancey (2004), and Turnbull et al. (2007).

Here we outline the KSB model as presented and extended by Ancey (2004), we will consider the two-dimensional motion of a cloud along a plane inclined at an angle $\tan \theta$ with respect to the horizontal. Figure 1.14 depicts a typical cloud entraining particles from the bed. In the following, $H$ denotes the cloud height, $L$ its length, $m$ its mass, $V$ its volume. The cloud velocity is $U = dx/dt$ but, since the body is deformable, the velocity varies inside the body. The front position is given by the abscissa $x_f$ while its velocity is $U_f = dx_f/dt$. The volume solid concentration is $\phi$ and it is assumed that the cloud is a homogeneous suspension of particles of density $\rho_p$ (no density stratification) in the ambient fluid of density $\rho_a$ and viscosity $\mu_a$. The bulk cloud density is then: \[ \bar{\rho} = \phi \rho_p + (1 - \phi) \rho_a. \]

Ahead of the front, there is a particle bed whose thickness is denoted by $h_n$, and which is made up of the same particles as the cloud. The apparent density of the layer is $\rho_s = \phi_m \rho_p + (1 - \phi_m) \rho_a$, where $\phi_m$ denotes the maximum random volume concentration of particles.

The surface area (per unit width) exposed to the surrounding fluid is denoted by $S$ and can be related to $H$ and $L$ as follows: $S = k_s \sqrt{HL}$, where $k_s$ is a shape factor. Here we assume that the cloud keeps a semi-elliptic form, whose aspect ratio $k = H/L$ remains constant during the cloud run when the slope is constant. We then obtain

\[ k_s = E(1 - 4k^2)/\sqrt{k}, \quad (1.46) \]

where $E$ denotes the elliptic integral function. Similarly, we can also express the volume $V$ (per unit width) as: $V = k_v HL$, where $k_v$ is another shape factor for a half ellipsis. Here we have

\[ k_v = \pi/4. \quad (1.47) \]

In the following, we will also need to use the volume, height, and length...
growth rates:

\[ \alpha_v = \frac{1}{\sqrt{V}} \frac{dV}{dx}, \quad \alpha_h = \frac{dH}{dx}, \quad \alpha_l = \frac{dL}{dx}. \quad (1.48) \]

Experimentally, it is easier to measure the growth rates by deriving the quantity at hand by the front abscissa instead of the mass center abscissa; we will refer to these rates as:

\[ \tilde{\alpha}_v = \frac{1}{\sqrt{V}} \frac{dV}{dx_f}, \quad \tilde{\alpha}_h = \frac{dH}{dx_f}, \quad \tilde{\alpha}_l = \frac{dL}{dx_f}. \quad (1.49) \]

Note that all these quantities are interrelated. For instance, using \( x = x_f - L/2 \), we find:

\[ \tilde{\alpha}_h = (dH/dx)(dx/dx_f) = \alpha_h(1 - \tilde{\alpha}_l/2). \]

Similarly, using the definition of \( k \) and \( k_v \), we obtain:

\[ \alpha_h = \frac{\alpha_v}{2} \sqrt{\frac{k}{k_v}} \quad \text{and} \quad \alpha_l = \frac{\alpha_v}{2\sqrt{kk_v}}. \quad (1.50) \]

The KSB model outlined here includes three equations: volume, mass, and momentum balances. The volume variations mainly result from the entrainment of the ambient, less dense fluid. To express the volume balance equation, the commonest assumption is to state that the volume variations come from the entrainment of the ambient fluid into the cloud and that the inflow rate is proportional to the exposed surface area and a characteristic
velocity $u_e$. This leads to the equation:

$$\frac{dV}{dt} = E_v S u_e,$$

(1.51)

where $E_v$ is the bulk entrainment coefficient and is a function of the Richardson number (1.1). According to the flow conditions, different expressions of $E_v$ have been drawn from experiments. Interestingly enough, the value of $E_v$ has been expressed very differently depending on whether the current is steady or unsteady. There is, however, no clear physical reason justifying this partitioning. Indeed, for most experiments, the currents were gradually accelerating and mixing still occurred as a result of the development of Kelvin-Helmholtz billows, thus very similarly to the steady case. This prompted Ancey (2004) to propose a new expression of the entrainment coefficient for clouds, which holds for both steady and slightly unsteady conditions: Ancey (2004) related $E_v$ (or $\alpha_v$) as a function of $Ri$ (instead of $\theta$ as done by previous authors): for $Ri \leq 1$, $\alpha_v = e^{-1.6Ri^2}$ while for $Ri > 1$, $\alpha_v = 0.2/Ri$.

The cloud mass can vary as a result of the entrainment of the surrounding fluid and/or the entrainment of particles from the bed. The former process is easily accounted for: during a short time increment $\delta t$, the cloud volume $V$ is increased by a quantity $\delta V$ mainly as a result of the air entrainment, thus the corresponding increase in the cloud mass is $\rho_a \delta V$. The latter process is less well known. In close analogy with sediment erosion in rivers and turbidity currents, Fukushima & Parker (1990) assumed that particles are continuously entrained from the bed when the drag force exerted by the cloud on the bed exceeds a critical value. This implies that the particle entrainment rate is controlled by the surface of the bed in contact with the cloud and the mismatch between the drag force and the threshold of motion. Here, since in extreme conditions the upper layers of the snow cover made up of new snow of weak cohesion can be easily entrained, it is reasonable to think that all the recent layer ahead of the cloud is incorporated into the cloud: when the front has traveled a distance $U_f \delta t$, where $U_f$ is the front velocity, the top layer of depth $h_n$ and density $\rho_s$ is entirely entrained into the cloud (see Fig. 1.14). The resulting mass variation (per unit width) is written: $\rho_s U_f h_n \delta t$. At the same time, particles settle with a velocity $v_s$. During the time step $\delta t$, all the particles contained in the volume $Lv_s \delta t$ deposit. Finally, by taking the limit $\delta t \to 0$, we can express the mass balance equation as follows:

$$\frac{dm}{dt} = \rho_a \frac{dV}{dt} + \rho_s U_f h_n - \phi \rho_s Lv_s,$$
where $m = \bar{\rho}V$ is the cloud mass. Usually the settling velocity $v_s$ is very low compared to the mean forward velocity of the front so that it is possible to ignore the third term on the right-hand side of the equation above. We then obtain the following simplified equation:

$$\frac{d\Delta \bar{\rho}V}{dt} = \rho_s U_f h_n.$$  \hfill (1.52)

The cloud undergoes the driving action of gravity and the resisting forces due to the ambient fluid and the bottom drag. The driving force per unit volume is $\bar{\rho}g \sin \theta$. Most of the time, the bottom drag effect plays a minor role in the accelerating and steady-flow phases but becomes significant in the decelerating phase (Hogg & Woods, 2001). Since we have set aside a number of additional effects (particle sedimentation, turbulent kinetic energy), it seems reasonable to also discard this frictional force. The action of the ambient fluid can be broken into two terms: a term analogous to a static pressure (Archimede’s theorem), equal to $\rho_a V g$, and a dynamic pressure. As a first approximation, the latter term can be evaluated by considering the ambient fluid as an inviscid fluid in an irrotational flow. On the basis of this approximation, it can be shown that the force exerted by the surrounding fluid on the half cylinder is $\rho_a V \chi \frac{dU}{dt}$, where

$$\chi = k$$  \hfill (1.53)

is called the \textit{added mass coefficient}. Since at the same time volume $V$ varies and the relative motion of the half cylinder is parallel to its axis of symmetry, we finally take: $\rho_a \chi \frac{d(UV)}{dt}$. Note that this parameter could be ignored for light interstitial fluids (e.g., air), whereas it has a significant influence for heavy interstitial fluids (basically water). Thus the momentum balance equation can be written as:

$$\frac{d(\bar{\rho} + \chi \rho_a)UV}{dt} = \Delta \bar{\rho}gV \sin \theta.$$  \hfill (1.54)

Analytical solutions can be found in the case of a Boussinesq flow ($\hat{\rho}/\rho_a \rightarrow 1$); for the other cases, numerical methods must be used. In the Boussinesq limit, since the final analytical solution is complicated, we only provide an asymptotic expression at early and late times. To simplify the analytical expressions, without loss of generality, here we take: $U_0 = 0$ and $x_0 = 0$ and we assume that the erodible snowcover thickness $h_n$ and density $\rho_s$ are constant. The other initial conditions are: at $t = 0$ and $x = 0$, $H = H_0$, $L = L_0$, $V_0 = k_v H_0 L_0$, and $\bar{\rho} = \bar{\rho}_0$. At short times, the velocity is independent
of the entrainment parameters and the initial conditions ($\bar{\rho}_0$ and $V_0$):

$$U \propto \sqrt{2g \sin \theta \frac{\Delta \rho_0}{\Delta \rho_0 + (1 + \chi) \rho_a}} \approx \sqrt{2g \sin \theta}$$  \hspace{1cm} (1.55)$$

where we used $\rho_a \ll \Delta \bar{\rho}_0$. This implies that the cloud accelerates vigorously in the first instants ($dU/dx \to \infty$ at $x = 0$), then its velocity grows more slowly. At long times for an infinite plane, the velocity reaches a constant asymptotic velocity that depends mainly on the entrainment conditions for flows in the air:

$$U_\infty \propto \sqrt{\frac{2gh_n(1 + \frac{\rho_0}{2}) \sin \theta \rho_a}{\alpha_v^2(1 + \chi) \rho_a}}$$  \hspace{1cm} (1.56)$$

Because of the slow growth of the velocity, this asymptotic velocity is reached only at very long times. Without particle entrainment, the velocity reaches a maximum at approximately $x_m^2 = (2\rho_0/3\rho_a)\alpha_v^{-2}V_0/(1 + \chi)$:

$$U_m^2 \approx \frac{4}{\sqrt{3}} \sqrt{\frac{\rho_0 g\sqrt{V_0} \sin \theta}{\rho_a \alpha_v \sqrt{1 + \chi}}}$$

then it decreases asymptotically as:

$$U \propto \sqrt{\frac{8\Delta \rho_0 g V_0 \sin \theta}{3\rho_a x \alpha_v^2(1 + \chi)}}$$  \hspace{1cm} (1.57)$$

In this case, the front position varies with time as:

$$x_f \propto (\rho_0 V_0 \sin \theta)^{1/3} t^{2/3}$$  \hspace{1cm} (1.58)$$

These simple calculations show the substantial influence of the particle entrainment on cloud dynamics. In the absence of particle entrainment from the bed, the fluid entrainment has a key role since it directly affects the value of the maximum velocity that a cloud can reach.

Here, we examine only the avalanche of 25 February 1999, for which the front velocity was recorded. In Fig. 1.15, we have reported the variation in the mean front velocity $U_f$ as a function of the horizontal downstream distance $y_f$: the dots correspond to the measured data while the curves represent the solution obtained by integrating Eqs. (13–16) numerically and by assuming that the growth rate coefficient depends on the overall Richardson number (solid line). For the initial conditions, we assume that $u_0 = 0$, $h_0 = 2.1 \text{ m}$, $l_0 = 20 \text{ m}$, and $\rho_0 = \rho_a = 150 \text{ kg/m}^3$. Due to the high path gradient between the origin and the elevation $z = 1800 \text{ m}$ ($y = 1250 \text{ m}$) we have considered that on average, the released snow layer $h_n$ is 0.7 m thick and is entirely entrained into the avalanche. Using $\alpha_v \propto R_i^{-1}$ for $R_i \gg 1$, we
apply the following relationship: for $Ri \leq 1$, $\alpha_v = e^{-1.6Ri^2}$ while for $Ri > 1$, we take $\alpha_v = 0.2/Ri$.

As shown in Fig. 1.15, the avalanche accelerated vigorously after the release and reached velocities as high as 80 m/s. The velocity variation in the release phase is fairly well described by the KSB model. The model predicts a bell-shaped velocity variation while field data provide a flatter velocity variation. The computed flow depth at $z = 1640$ m is approximately 60 m which is consistent with the value estimated from the video tapes. In order to evaluate the sensitivity of the simulation results, we examined different values of the erodible mass. In Figure 1.15, we have reported the comparison between field data and computations made with three different assumptions: $\rho_s h_n = 50$, 105, or 150 kg/m$^2$. It can be seen that there is no significant variation in the computed velocities in the accelerating phase, but both the maximum velocity and the position at which the maximum velocity is reached depend on the $\rho_s h_n$ value. By increasing the erodible mass per unit surface from 50 to 150 kg/m$^2$, the maximum velocity is increased from 69 m/s to 105 m/s, i.e., by a factor of 1.5. Note that the dependence of the maximum velocity on the snowcover thickness is consistent with field measurements: for instance, the avalanche of 10 February 1999 was approximately half as large in terms of deposited volume as the avalanche of 25 February 1999, and its maximum velocity was 25% lower than the maximum velocity recorded on 25 February 1999. This result is of great importance in engineering applications since it means that the maximum velocity and,
thereby, the destructive power of a powder-snow avalanche primarily result from the ability to entrain snow from the snowcover when the avalanche descends.

1.6.2 Depth-averaged equations

An airborne avalanche is a very turbulent flow of a dilute ice–particle suspension in air. It can be considered as a one-phase flow as a first approximation. Indeed, the Stokes number defined as the ratio of a characteristic time of the fluid to the relaxation time of the particles is low, implying that particles adjust quickly to changes in the air motion. At the particle scale, fluid turbulence is high enough to strongly shake the mixture since the particle size is quite small. To take into account particle sedimentation, authors generally consider airborne avalanches as turbulent stratified flows. Thus, contrary to flowing avalanches, bulk behavior is well identified in the case of airborne avalanches. The main differences between the various models result from the different boundary conditions, use of the Boussinesq approximation, and the closure equations for turbulence. Parker et al. (1986) developed a complete depth-averaged model for turbidity currents. The equations of motion proposed by these authors are more complicated than the corresponding set for dense flows presented in § 1.5.1, since they include additional equations arising from the mass balance for the dispersed phase, the mean and turbulent kinetic energy balances, and the boundary conditions related to the entrainment of sediment and surrounding fluid:

\[
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = E_a u ,
\]

\[
\frac{\partial (C h)}{\partial t} + \frac{\partial (h u C)}{\partial x} = v_s E_s - v_s c_b ,
\]

\[
\frac{\partial h u}{\partial t} + \frac{\partial (hu^2)}{\partial x} = R C g h \sin \theta - \frac{1}{2} R g \frac{\partial (Ch^2)}{\partial x} - u_s^2 ,
\]

\[
\frac{\partial h K}{\partial t} + \frac{\partial hu K}{\partial x} = \frac{1}{2} E_a u^3 + u_s^2 u - \varepsilon_0 h - \frac{1}{2} E_a u R C g h - \frac{1}{2} R g h v_s (2C + E_s - c_b) ,
\]

where \( u \) is the mean velocity, \( h \) the flow depth, \( K \) the mean turbulent kinetic energy, \( C \) the mean volume concentration (ratio of particle volume to total volume), \( E_a \) a coefficient of entrainment of surrounding fluid into the current, \( v_s \) the settlement velocity, \( E_s \) a coefficient of entrainment of particles from the bed into the current, \( c_b \) the near-bed particle concentration, \( R \) the
specific submerged gravity of particles (ratio of buoyant density to ambient fluid density), $u^2_*$ the bed shear velocity, and $\varepsilon_0$ the depth-averaged mean rate of dissipation of turbulent energy due to viscosity. The main physical assumption in Parker et al.’s model is that the flow is considered as a one-phase flow from a momentum point of view but treated as a two-phase flow concerning the mass balance. Equation (1.59) states that the total volume variation results from entrainment of surrounding fluid. In (1.60), the variation in the mean solid concentration is due to the difference between the rate of particles entrained from the bed and the sedimentation rate. Equation (1.61) is the momentum balance equation: the momentum variation results from the driving action of gravity and the resisting action of bottom shear stress; depending on the flow depth profile, the pressure gradient can contribute either to accelerate or decelerate the flow. Equation (1.62) takes into account the turbulence expenditure for the particles to stay in suspension. Turbulent energy is supplied by the boundary layers (at the flow interfaces with the surrounding fluid and the bottom). Turbulent energy is lost by viscous dissipation ($\varepsilon_0 h$ in (1.62)) as well as by mixing the flow (fourth and fifth terms in (1.62)) and maintaining the suspension against sedimentation flow mixing (last term on the right-hand side of (1.62)).

Although originally devoted to submarine turbidity currents, this model has been applied to airborne avalanches, with only small modifications in the entrainment functions (Fukushima & Parker, 1990). A new generation of powder-snow avalanche models has recently appeared (Hutter, 1996). Some rely on the numerical resolution of local equations of motion, including a two-phase mixture approximation and closure equations, usually a $k-\varepsilon$ model for turbulence (Hutter, 1996). Other researchers have tried to establish the relation existing between a dense core and an airborne avalanche because they think that, most often, a powder-snow avalanche is tightly related to a denser part that supplies the airborne part with snow (Eglit, 1983; Nazarov, 1991; Issler, 1998). Though these recent developments are undoubtedly a promising approach to modelling powder-snow avalanches, their level of sophistication contrasts with the crudeness of their basic assumptions as regards the momentum exchanges between phases, turbulence modification due to the dispersed phase, and so on. At this level of our knowledge of physical and natural processes, it is of great interest to continue to use simple models and to fully explore what they can describe and explain.
Notes


References


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References


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