Synchronization in Coupled Complex Systems

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• Introduction
• Complex synchronization in simple geometry
  – Phase-coherent complex systems
  – Applications
• Synchronization in non-phase coherent systems: phase and/or generalized synchronization
  - Concepts of curvature and recurrence
  - Applications
• Conclusions
Greek origin:

Σύγχρονος – sharing a common property in time
Nonlinear Sciences

Start in 1665 by Christiaan Huygens:

Discovery of phase synchronization, called sympathy
Huygens´-Experiment
• Christiaan Huygens:

Pendulum clocks hanging at the same wooden beam (half-timber house)

It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the last bit from each other...Further, if this agreement was disturbed by some interference, it reestablished itself in a short time...after a careful examination I finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible (Huygens, 1673)
Modern Example: Mechanics

London’s Millenium Bridge

- pedestrian bridge
- 325 m steel bridge over the Themse
- Connects city near St. Paul’s Cathedral with Tate Modern Gallery

Big opening event in 2000 -- movie
Bridge Opening

- Unstable modes always there
- Mostly only in vertical direction considered
- Here: extremely strong unstable lateral Mode – If there are sufficient many people on the bridge we are beyond a threshold and synchronization sets in

(Kuramoto-Synchronizations-Transition, book of Kuramoto in 1984)
Stabilisierung nachträglich erreicht

GERB Schwingungsisolierungen GmbH, Berlin/Essen
Examples: Sociology, Biology, Acoustics, Mechanics

- Hand clapping (common rhythm)
- Ensemble of doves (wings in synchrony)
- Mexican wave
- Menstruation (e.g. female students living in one room in a dormitory)
- Organ pipes standing side by side – quenching or playing in unison (Lord Rayleigh, 19th century)
- Fireflies in south east Asia (Kämpfer, 17th century)
- Crickets and frogs in South India
Synchronization in populations

![Graph and Image Description]

- **Amount of sync**: fraction of population running at the same speed.
- **Phase transition**: from incoherent to perfect sync.
- **Amount of homogeneity in population**:
  - Very diverse
  - Moderately diverse
  - Clones
Necessary Conditions for Synchronization

• **Two Oscillators** (or more; best: self-sustaining)
• **Coupling:** Master – Slave, or mutually coupled
• **Starting:** (slightly) different systems
  (initial conditions, internal frequencies)
• **Goal:** becoming identical in a main property or sharing some important behaviour due to forcing or interaction
  (becoming identical, adjusting their phases…)
Types of Synchronization in Complex Processes

- **phase synchronization**
  phase difference bounded, a zero Lyapunov exponent becomes negative (phase-coherent)
  (Rosenblum, Pikovsky, Kurths, 1996)

- **generalized synchronization**
  a positive Lyapunov exponent becomes negative, amplitudes and phases interrelated
  (Rulkov, Sushchik, Tsimring, Abarbanel, 1995)

- **complete synchronization** (Fujisaka, Yamada 1983)
Most systems not simply periodic

→ Synchronization in complex (non-periodic) systems

Interest in **Phase Synchronization**

How to retrieve a phase in complex dynamics?
Rössler Oscillator – 2D Projection

Phase-coherent (projection looks like a smeared limit cycle, low diffusion of phase dynamics)
Phase dynamics in periodic systems

• Linear increase of the phase

\[ \phi (t) = t \omega \]

\[ \omega = 2 \pi / T \] – frequency of the periodic dynamics

\[ T \] – period length

⇒ \( \phi (t) \) increases \( 2 \pi \) per period

\[ d \phi (t) / d t = \omega \]
Phase Definitions

Analytic Signal Representation (Hilbert Transform)

\[ \psi(t) = s(t) + j\tilde{s}(t) = A(t)e^{j\phi(t)} \]

\[ \tilde{s}(t) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} \, d\tau \]

Direct phase

\[ \phi(t) = \arctan\left(\frac{y(t)}{x(t)}\right) \]

Phase from Poincare’ plot

\[ \phi(t) = 2\pi k + 2\pi \frac{t - \tau_k}{\tau_{k+1} - \tau_k} \quad (\tau_k < t < \tau_{k+1}) \]

Dennis (De´nes) Gabor
1900-1979

Electrical Engineer

Nobel Price
Physics 1971
Hilbert transform for periodic signals
Hilbert transform for chaotic signals

(a) $x(t), A(t)$

(b) $\phi(t)$

(c) $\phi(t) - \omega_0 \tau$
Phase for coherent chaotic oscillators

Phase dynamics and phase synchronization phenomena very similar in periodic and phase-coherent chaotic systems, e.g. one zero Lyapunov exponent becomes negative.
Synchronization due to periodic driving

\[ \dot{x} = -\omega y - z + E \sin(\Omega_c t) \]

\[ \dot{y} = \omega x + ay , \]

\[ \dot{z} = f + z(x - c) \]
Synchronization due to periodic driving

Fig. 3.5. Stroboscopic plot of the Rössler system state \((x, y)\) (filled cycles) at each period of the driving signal (Eq. (3.4)). The dotted background is the unforced chaotic attractor. (a) \(E = 0.15, \Omega_e = 1.0\), phase is synchronized. (b) \(E = 0.15, \Omega_e = 1.02\), phase is not synchronized.
Synchronization of two coupled non-identical chaotic oscillators

\[
\begin{align*}
\dot{x}_{1,2} &= -\omega_{1,2} y_{1,2} - z_{1,2} + C(x_{2,1} - x_{1,2}), \\
\dot{y}_{1,2} &= \omega_{1,2} x_{1,2} + ay_{1,2}, \\
\dot{z}_{1,2} &= f + z_{1,2}(x_{1,2} - c),
\end{align*}
\]

Phases are synchronized \quad BUT \quad Amplitudes almost uncorrelated

(a) Phases synchronization vs time with different coupling strengths C.

(b) Phase space plot showing the trajectory of the system.
Synchronization of two coupled chaotic oscillators

Synchronization region

$C$ – coupling strength
$\Delta \omega$ – parameter mismatch
Two coupled non-identical Roessler oscillators

\[ \phi = \arctan(y/x), \quad A = (x^2 + y^2)^{1/2}, \]

we get

\[ \dot{A}_{1,2} = aA_{1,2} \sin^2 \phi_{1,2} - z_{1,2} \cos \phi_{1,2} + C(A_{2,1} \cos \phi_{2,1} \cos \phi_{1,2} - A_{1,2} \cos^2 \phi_{1,2}), \]

\[ \dot{\phi}_{1,2} = \omega_{1,2} + a \sin \phi_{1,2} \cos \phi_{1,2} + z_{1,2}/A_{1,2} \sin \phi_{1,2} - C(A_{2,1}/A_{1,2} \cos \phi_{2,1} \sin \phi_{1,2} - \cos \phi_{1,2} \sin \phi_{1,2}), \]

\[ \dot{z}_{1,2} = f - cz_{1,2} + A_{1,2}z_{1,2} \cos \phi_{1,2}. \]

Equation for the slow phase \( \theta \): \[ \phi_{1,2} = \omega_0 t + \theta_{1,2}. \]

Averaging yields (Adler-like equation, phase oscillator):

\[ \frac{d}{dt}(\theta_1 - \theta_2) = 2\Delta \omega - \frac{C}{2} \left( \frac{A_2}{A_1} + \frac{A_1}{A_2} \right) \sin(\theta_1 - \theta_2). \]
Fixed point solution (by neglecting amplitude fluctuations)

\[ \theta_1 - \theta_2 = \arcsin \frac{4\Delta\omega A_1 A_2}{C(A_1^2 + A_2^2)} \]

Fixed point stable (synchronization) if coupling is larger than

\[ C_{PS} = 4\Delta\omega A_1 A_2/(A_1^2 + A_2^2). \]
Imperfect Phase Synchronization

Lorenz system:
Periodically forced

\[ \begin{align*}
\dot{x} &= 10(y - x), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - 2.667z + E \cos(\Omega t)
\end{align*} \]
Unstable Periodic Orbits – usual Lorenz system
Synchronization regions of UPOs in the usual Lorenz system

Solid lines – 1:1 synchronization
Dashed line – 14:15 synchronization
Dotted line – 18:20 synchronization
Phase „jumps“ in the forced usual Lorenz model
Application

Basic Problem: Phase Jumps
Problems with phase jumps

- Phase synchronization: difference of phases of two (more) subsystems is bounded
- Phase jumps of $\pm 2\pi$ occur due to
  - at the borderline of synchro region
  - influence of noise (Stratonovich)
  - broad variety of unstable periodic orbits, as in the Lorenz system (deterministic effect)
  
$\Rightarrow$ imperfect phase synchronization

Cyclic relative phase

• How to consider this problem?
• Phase synchronization in a statistical sense
• Instead of the strong condition

\[ |n\phi_1(t) - m\phi_2(t) - \delta| < \text{const} \]

We consider the cyclic relative phase

\[ \Psi_{n,m} = \varphi_{n,m} \mod 2\pi \]

And analyze the frequency distribution of the cyclic relative phase
Fig. 3. Relative phase $\varphi_{1,1} = \phi_1 - \phi_2$ and distribution of $\Psi_{1,1} = \varphi_{1,1} \mod 2\pi$ for the case of uncoupled (a,b) and coupled (c,d) non-identical chaotic systems perturbed by noise. The horizontal line in (c) corresponds to the absence of noise: in this case the phase difference fluctuates around some constant value due to influence of chaotic amplitudes. These fluctuations are rather small (barely seen in this scale), and no phase slips are observed; this fact is explained by the high phase coherence properties of the Rössler attractor. In contrast to the noisy case, here we observe both frequency and phase locking.
The synchronization index based on the Shannon entropy $S$ of the phase difference distribution [49]. Having an estimate $p_k$ of the distribution of $\Psi_{n,m}$, we define the index $\rho$ as

$$\rho_{n,m} = \frac{S_{\text{max}} - S}{S_{\text{max}}} ,$$

where $S = -\sum_{k=1}^{N} p_k \ln p_k$, and the maximal entropy is given by $S_{\text{max}} = \ln N$; $N$ is the number of bins and $p_k$ is the relative frequency of finding $\Psi_{n,m}$ within the $k$-th bin. Due to the normalization used,

$$0 \leq \rho_{n,m} \leq 1 ,$$

whereas $\rho_{n,m} = 0$ corresponds to a uniform distribution (no synchronization) and $\rho_{n,m} = 1$ corresponds to a distribution localized in one point ($\delta$-function). Such distribution can be observed only in the ideal case of phase locking of noise-free quasilinear oscillators.
Efficiency of these criteria

Fig. 9. Comparison of quantitative measures of synchronization using the simulated data from two coupled Rössler oscillators (Eq. (10)). Transition to synchronous state takes place when the difference of frequencies of two oscillators vanishes with increase of coupling coefficient $\varepsilon$ (a). Three 1 : 1 synchronization indices are shown as the functions of $\varepsilon$ (b).
(Phase) Synchronization – good or bad???

Context-dependent
Applications in various fields

Lab experiments:

- Electronic circuits (Parlitz, Lakshmanan, Dana...)
- Plasma tubes (Rosa)
- Driven or coupled lasers (Roy, Arecchi...)
- Electrochemistry (Hudson, Gaspar, Parmananda...)
- Controlling (Pisarchik, Belykh)
- Convection (Maza...)

Natural systems:

- Cardio-respiratory system (Nature, 1998...)
- Parkinson (PRL, 1998...)
- Epilepsy (Lehnertz...)
- Kidney (Mosekilde...)
- Population dynamics (Blasius, Stone)
- Cognition (PRE, 2005)
- Climate (GRL, 2005)
- Tennis (Palut)
Fig. 7. Stabilograms of a neurological patient for EO (a), EC (b), and AF (c) tests. The upper panels show the relative phase between two signals $x$ and $y$ that are deviation of the center of pressure in anterior–posterior and lateral direction, respectively. During the last 50s of the first test and the whole second test the phases are perfectly locked. No phase entrainment is observed in the AF test.

EO – eyes open, EC – eyes closed, AF – eyes open and visual feedback
Analysis of posturographic measurements of balance

Hypothesis:
Is there a relationship between cognition and motorics? Ability to control posture and school success
Application:

Cardiovascular System
FIG. 1. Short segments of an electrocardiogram with the R peaks marked (a) and of a respiratory signal (b); both signals are in arbitrary units.

Analysis technique: **Synchrogram**
Figure 1 Analysis of cardiorespiratory cycles. a, Cardiorespiratory synchronogram, showing the transition (red) from 5:2 frequency locking (black) to 3:1 phase locking (blue). Each point shows the normalized relative phase of a heartbeat within two adjacent respiratory cycles \( \psi(t_k) = \frac{(\phi(t_k) \mod 4\pi)}{2\pi} \). b, Number of heartbeats within two adjacent respiratory cycles. c, Histogram of phases. The six horizontal stripes in the blue region of the CRS result in six well-pronounced peaks in the distribution of phases. d, Autocorrelation function of phases \( R_\psi(\tau) = \frac{\sum_i (\psi(t_i) - \langle \psi \rangle)(\psi(t_{i+\tau}) - \langle \psi \rangle)}{\sum_i (\psi(t_i) - \langle \psi \rangle)^2} \). The coloured curves correspond to respective regions.
Fig. 23. A transient epoch within the data of subject A confirms the existence of synchronization. The periods of cardiac (R-R) and respiratory cycles (T) are shown in (a) and (b), respectively. After a short epoch of non-synchronous behavior (1150–1200s) the frequencies of heart rate and respiration change, probably due to influence of a certain control mechanism, and become locked, i.e., $f_r/f_h \approx 1/3$. In the next 50s we observe that, although both frequencies decrease, this ratio remains almost constant (c). This means that one of the systems follows the other one, i.e., synchronization takes place. 3 : 1 phase locking is also clearly seen from CRS (d).
Cardiorespiratory Synchronisation during Sleep

NREM

REM

Beat-to-beat intervals

Respiration
Cardiorespiratory Synchronisation during Sleep

Phil. Trans Roy Soc A, 2009

5:1 synchronization during NREM
Application of synchronization analysis: Mother-Fetus System

Magnetocardiography
Testing the foetal–maternal heart rate synchronization via model-based analyses


Figure 2. (a–d) Synchrograms of the data shown in figure 1 for maternal cycles $m=1, \ldots, 4$. The detected epochs of synchronization (see text) are marked by bold points. The synchronization ratio (b) $n : m=3 : 2$ and (c) $n : m=5 : 3$ were found. The other ratios ($1 : 1, 2 : 1, 5 : 2, 4 : 3, 5 : 4$ and $7 : 4$) were not detected. For instance, detected $6 : 4$ synchronization is not marked because it is twice the $3 : 2$ ratio.
Magnetocardiogram (MCG) Data – Paced Breathing

• 6 pregnant women, aged 33 +/- 4 years
  • 34th – 40th week of gestation

• 6 consecutive 5 min simultaneous fetal and maternal MCGs for the sequence:
  spontaneous, 15 cpm, 10 cpm, 20 cpm, 12 cpm, spontaneous breathing of mother;

  2-3 min pause between (40 min)
Influence of maternal paced breathing on the connections of the fetal-maternal cardiac systems.

Distribution of the synchronization epochs (SE) over the maternal beat phases with respect to the $n:m$ combinations 3:2 (top), 4:3 (middle) and 5:3 (bottom) in the different respiratory conditions.

Special test statistics: twin surrogates

van Leeuwen, Romano, Thiel, Wessel, Kurths, PNAS 106, 13661 (2009) (+ commentary)
Synchronization in more complex topology

- Systems are often **non-phase-coherent** (e.g. funnel attractor – much stronger phase diffusion)
- How to study phase dynamics there?
- 1st Concept: **Curvature**
  

\[
\phi = \arctan \frac{\dot{y}}{\dot{x}}.
\]
Roessler Funnel – Non-Phase coherent
Phase basing on curvature

curve \vec{r}_1 = (u, v) the angle velocity at each point is

\[ \nu = \frac{ds}{dt} / R, \]

where

\[ ds/dt = \sqrt{\dot{u}^2 + \dot{v}^2} \]

is the speed along the curve and

\[ R = (\dot{u}^2 + \dot{v}^2)^{3/2} / [\ddot{v}\dot{u} - \dddot{u}] \]

is the radius of the curvature. If \( R > 0 \) at each point, then

\[ \nu = \frac{d\phi}{dt} = \frac{\ddot{v}\dot{u} - \dddot{u}}{\dot{u}^2 + \dot{v}^2}, \]

is always positive and therefore the variable \( \phi \) defined as

\[ \phi = \int \nu dt = \arctan \frac{\dot{v}}{\ddot{u}}, \]
FIG. 1: Upper panel (a,b,c): projections of the attractors of the Rössler systems (1) onto the plane \((x, y)\); middle panel: (d,e,f): projections onto \((\dot{x}, \dot{y})\); lower panel (g,h,i): distribution of the return times \(T\). The parameters are \(\omega = 0.98\) and \(a = 0.16\) (a,d,g), \(a = 0.22\) (b,e,h) and \(a = 0.28\) (c,f,i).
Application

Climatology
Monsoon-Data

Rainfall [mm/month] vs. Time [years]
Teleconnections

• (Weak) Connections of meteorological conditions/ regimes between largely distant regions

• Examples: NAO – El Nino
  El Nino – Indian Monsoon
How to study such interactions?

Concept of Synchronization
Application: El Niño vs. Indian monsoon

- El Niño/Southern Oscillation (ENSO) – self-sustained oscillations of the tropical Pacific coupled ocean-atmosphere system
- Monsoon - oscillations driven by the annual cycle of the land vs. Sea surface temperature gradient
- ENSO could influence the amplitude of Monsoon – Is there phase coherence?
- Monsoon failure coincides with El Niño

(Maraun, Kurths, Geophys Res Lett (2005))
Figure 1. Section of the NINO3 (upper graph) and AIR anomalies (lower graph) time series. The dotted lines depict the raw data, the solid lines show the low-pass filtered data used for the further analysis.
Figure 2. (a) Embedding of low-pass filtered NINO3 time series by Hilbert transformation. Many oscillations are not centered around a common center. (b) The same, but for the time derivative of the NINO3 time series. All pronounced oscillations circle around the origin.
Phase coherence between El Niño and Indian monsoon

Figure 5. Phase difference of ENSO and Monsoon (black). Grey shading marks intervals of jointly well defined phases. 1886-1908 and 1964-1980 (I): plateaus indicate phase coherence. 1908-1921, 1935-1943 and 1981-1991 (II): Monsoon oscillates with twice the phase velocity of ENSO. During these intervals, both systems exhibit distinct oscillations (NINO3 time series, upper graph). 1921-1935 and 1943-1963: phases are badly defined, both processes exhibit irregular oscillations of low variance (upper graph). Lower graph shows volcanic radiative forcing index (VRF).
Figure 6. Histogram of phase differences for the two phase coherent intervals (a) 1886-1908, (b) 1964-1980. Both diagrams show peaks between $\pi/2$ and $\pi$, reflecting that ENSO and Monsoon are anti-correlated.
Wavelet Analysis

Thick black curve – statistically significant

Thin black curve – not-correct statistical significance region
Directionality Analysis based on Granger Causality

• Linear (AR) and nonlinear (polynomials) dependences

• Result: bidirectional alternating dependence (Geoph. Res. Lett. 2011)

ENSO $\Rightarrow$ Monsoon 1890-1920 and 1950-1980
Monsoon $\Rightarrow$ ENSO 1917-1927 and 1980-1990
Granger causality

- ENSO $\rightarrow$ Monsoon

\[ x_1(t) = a_{1,1}x_1(t-1) + b_{1,1}x_2(t-1) + c_{1,1}x_1^2(t-1)x_2(t-1) + c_{1,2}x_2^3(t-1) + \eta_1(t), \]

- Monsoon $\rightarrow$ ENSO

\[ x_2(t) = a_{2,1}x_2(t-1) + a_{2,5}x_2(t-5) + b_{2,1}x_1(t-1) + b_{2,2}x_1(t-2) + b_{2,3}x_1(t-3) + \eta_2(t), \]
EU FP7 Project

**Basic idea:** couple different models during simulation based on synchronization

Methods from nonlinear dynamics and machine learning

Application to large climate models:
Frank Selten (KNMI) & Noel Keenlyside (Uni Bergen)
The supermodeling approach

Combine individual models $\mu$ into one supermodel by inserting nonnegative connections between the model equations:

$$\dot{x}^i_\mu = f^i_\mu(x_\mu) + \sum_\nu C^i_{\mu\nu}(x^i_\nu - x^i_\mu)$$

Solution of the supermodel defined to be the average of the $M$ coupled imperfect models:

$$x_{\text{sumo}}(C, t) = \frac{1}{M} \sum_\mu x_\mu(C, t)$$

Cost function defined by the accumulated error of $K$ runs:

$$E(C) = \frac{1}{K\Delta} \sum_{i=1}^{K} \int_{t_i}^{t_i+\Delta} |x_{\text{sumo}}(C, t) - x_{gt}(t)|^2 \gamma^t dt$$
The synchronization of chaotic systems

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Synchronization in complex networks

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Synchronization in Oscillatory Networks

The formation of collective behavior in large ensembles or networks of coupled oscillatory elements is one of the oldest and most fundamental aspects of dynamical systems theory. Potential and present applications span a vast spectrum of fields ranging from physics, chemistry, geoscience, through life- and neurosciences to engineering, the economic and the social sciences. This work systematically investigates a large number of oscillatory network configurations that are able to describe many real systems such as electric power grids, lasers or the heart muscle – to name but a few.

This book is conceived as an introduction to the field for graduate students in physics and applied mathematics as well as being a compendium for researchers from any field of application interested in quantitative models.